

# BERTINI THEOREM FOR NORMALITY ON LOCAL RINGS IN MIXED CHARACTERISTIC (APPLICATIONS TO CHARACTERISTIC IDEALS)

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**ABSTRACT.** In this article, we prove a strong version of local Bertini theorem for normality on local rings in mixed characteristic. The main result asserts that a generic hyperplane section of a normal, Cohen-Macaulay, and complete local domain of dimension at least 3 is normal. Applications include the study of characteristic ideals attached to torsion modules over Noetherian normal domains, which is fundamental in the study of Euler system theory over normal domains and Iwasawa main conjectures.

## 1. INTRODUCTION

The classical Bertini theorem says that a generic hyperplane section of a smooth complex projective variety is smooth. A (local) Bertini theorem may be stated for a local ring  $(R, \mathfrak{m}, \mathbf{k})$  as follows. Let  $\mathbf{P}$  be a ring-theoretic property (e.g. regular, reduced, normal, seminormal and so on). Then if  $R$  is  $\mathbf{P}$  and  $x \in \mathfrak{m}$  is a nonzero divisor, then is it true that  $R/xR$  is so for a generic choice of  $x$ ? A local Bertini theorem (in a slightly weak form) was first raised by Grothendieck ([4], Exposé XIII, Conjecture 2.6) and was proved by Flenner [1]. It was, however, found later that his proof had a small gap in the mixed characteristic case, and the proof was finally remedied by Trivedi [13]. In this article, for an ideal  $I \subseteq R$ , we denote by  $D(I)$  the set of all primes of  $R$  which do not contain  $I$ , and by  $V(I)$  the complement of  $D(I)$  in  $\text{Spec } R$ . Before stating our main theorems, let us recall the following result from [1]:

**Theorem 1.1** (Flenner-Trivedi). *Let  $(R, \mathfrak{m})$  be a local Noetherian ring and let  $I \subseteq \mathfrak{m}$  be an ideal. Assume that  $Q$  is a finite subset of  $D(I)$ . Then there exists an element  $x \in I$  such that:*

- (1)  $x \notin \mathfrak{p}^{(2)}$  for all  $\mathfrak{p} \in D(I)$ ;
- (2)  $x \notin \mathfrak{p}$  for all  $\mathfrak{p} \in Q$ ;

for  $\mathfrak{p}^{(n)} := \mathfrak{p}^n R_{\mathfrak{p}} \cap R$ , the  $n$ -th symbolic power ideal of  $\mathfrak{p}$ .

We make a remark on the second symbolic power of ideals. Let  $(R, \mathfrak{m})$  be a local ring, let  $x \in \mathfrak{p}$  be a nonzero divisor and let  $x \notin \mathfrak{p}^{(2)}$  for a prime ideal  $\mathfrak{p} \subseteq R$ . Then  $R_{\mathfrak{p}}$  is regular

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if and only if  $R_{\mathfrak{p}}/xR_{\mathfrak{p}}$  is regular. Many ring-theoretic properties such as regular, normal, reduced can be verified at the localization  $R_{\mathfrak{p}}$ , which is the reason why we require  $x \notin \mathfrak{p}^{(2)}$  (but not merely  $x \notin \mathfrak{p}^2$ ), which is equivalent to the condition:  $x \notin \mathfrak{p}^2R_{\mathfrak{p}} \cap R$ . A strong version of local Bertini theorem similar to our main theorem below was already proved for local rings containing a field in [1]. To extend his result to the mixed characteristic case, we need to introduce some new ideas. Theorem 1.1 does not suffice for our purpose, since it only assures us that there is an element  $x \in \mathfrak{m}$  for which  $R/xR$  is normal.

Here is our notation which will be used throughout: We denote by  $\mathbb{P}^n(S)$  (resp.  $\mathbb{A}^n(S)$ ) the projective space (resp. an affine space) for a commutative ring  $S$ . In case when  $S$  is a field or a discrete valuation ring, we will consider these spaces as sets of points in the ring  $S$  endowed with some topology (see discussions in § 2). In these cases, every point  $a = (a_0 : \cdots : a_n) \in \mathbb{P}^n(S)$  is normalized so that  $a_i$  is a unit for some  $i$ . This implies that, for a complete discrete valuation ring  $A$  with residue field  $\mathbf{k}$ , we have a *specialization map*  $\mathrm{Sp}_A : \mathbb{P}^n(A) \rightarrow \mathbb{P}^n(\mathbf{k})$ , which is well-defined. Let us now state our main theorem (see Theorem 4.4 together with Theorem 4.3 in § 4).

**Main Theorem 1** (Local Bertini Theorem). *Let  $(R, \mathfrak{m}, \mathbf{k})$  be a complete local domain of mixed characteristic  $p > 0$  and suppose the following conditions:*

- (1) *let  $A \rightarrow R$  be a coefficient ring map for a complete discrete valuation ring  $(A, \pi_A)$ ;*
- (2) *let  $x_0, x_1, \dots, x_d$  be a fixed set of minimal generators of  $\mathfrak{m}$ ;*
- (3)  *$R$  is normal, of depth  $R \geq 3$ , and the residue field  $\mathbf{k}$  is infinite.*

*Then there exists a Zariski dense open subset  $U \subseteq \mathbb{P}^d(\mathbf{k})$  satisfying the following properties. For any  $a = (a_0 : \cdots : a_d) \in \mathrm{Sp}_A^{-1}(U)$ , the quotient  $R/\mathbf{x}_a R$  is a normal domain of mixed characteristic  $p > 0$ , where we put*

$$\mathbf{x}_a := \sum_{i=0}^d a_i x_i.$$

We will also discuss a version of the above theorem for the case when the residue field is finite at the end of § 4. This theorem allows us to find sufficiently many normal local domains of mixed characteristic as specializations. The above theorem does not tell us how to find  $U$ , but we will show how to find  $U$  in Example 4.9. It is worth pointing out that if  $\dim R = 2$ , the local Bertini theorem fails due to a simple reason. In fact, if the quotient  $R/yR$  is normal, then it is a discrete valuation ring, so  $R$  must be regular. By Cohen structure theorem, there is a surjection  $A[[z_0, \dots, z_d]] \twoheadrightarrow R$ , where  $d + 1$  is the number of minimal generators of  $\mathfrak{m}$ , so that the minimal generators of  $\mathfrak{m}$  are just the image of  $z_0, z_1, \dots, z_d$  under this surjection. Related to our main result, we remark the following converse which might be known to experts, but we find no standard reference:

*Remark 1.2.* Let  $(R, \mathfrak{m})$  be a local ring and  $y \in \mathfrak{m}$  a nonzero divisor. If  $R/yR$  is normal, then so is  $R$ . This will be done by showing that  $R$  satisfies both Serre's  $(R_1)$  and  $(S_2)$ .

Let  $P$  be a prime such that  $\text{ht } P = 1$ . If  $P = yR$ , then  $R_P$  is regular. If  $P \neq yR$ , then we can find a height 2 prime  $Q$  containing both  $y$  and  $P$ , the  $(R_1)$  condition on  $R/yR$  assures that the image  $\overline{Q}$  of  $Q$  in  $(R/yR)_{\overline{Q}}$  is principal, which implies that  $Q$  is generated by at most 2 elements. Hence  $R_P$  is regular.

Next, let  $P$  be a prime such that  $\text{ht } P \geq 2$ . If  $y \in P$ , we have  $\text{depth}_P R \geq 2$ , because  $y$  is a nonzero divisor and  $P \notin \text{Ass}(yR)$ . If  $y \notin P$  and  $Q$  is a minimal prime over  $P + yR$ , then we have  $\text{depth}_Q R/yR \geq 2$ , due to the  $(S_2)$  condition on  $R/yR$ . Thus  $\text{depth}_Q R \geq 3$ . Since  $\text{ht } Q/P = 1$ , it follows that  $\text{depth}_Q R \leq \text{depth}_P R + 1$ , say  $\text{depth}_P R \geq 2$ . This completes the proof.

In § 7, we prove some basic results on characteristic ideals over general normal domains. As an application, we prove an important result (see Theorem 8.7 in § 8) on characteristic ideals attached to torsion modules over normal domains via local Bertini theorem. This result will be crucial in a forthcoming paper [9], where we plan to study certain torsion modules arising from Iwasawa theory as developed in [8] (for example, those torsion modules arising as the Pontryagin dual of Selmer groups associated with two-dimensional Galois representations of certain type with values in a complete local ring with finite residue field). In the present article, we often make use of a lemma of “generic freeness” for which we refer to ([5], Theorem 24.1).

## 2. SPECIALIZATION MAP AND THE TOPOLOGY FOR BERTINI-TYPE THEOREMS

Let us discuss the specialization map on the projective spaces. Let  $A$  be a complete discrete valuation ring and let  $a := (a_0 : \cdots : a_n) \in \mathbb{P}^n(A)$ . Then we may normalize  $a$  so that  $a = (b_0 : \cdots : b_n) \in \mathbb{P}^n(A)$  and some  $b_i$  is a unit of  $A$ . So we may think of the projective space as

$$\mathbb{P}^n(A) = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n(A) \mid |a_i| = 1 \text{ for some } i\}$$

for the normalized valuation  $|\cdot| : A \rightarrow \mathbb{R}_{\geq 0}$ . Let  $\mathbf{k}$  be the residue field of  $A$ . Then reducing  $a$  to a point  $\bar{a} = (\bar{b}_0 : \cdots : \bar{b}_n) \in \mathbb{P}^n(\mathbf{k})$ , we have constructed the map:

$$\text{Sp}_A : \mathbb{P}^n(A) \rightarrow \mathbb{P}^n(\mathbf{k}),$$

called the *specialization map*. This map does not depend on the choice of the representative of  $a \in \mathbb{P}^n(A)$  as above. The set  $\mathbb{P}^n(\mathbf{k})$  is endowed with the Zariski topology, while  $\mathbb{P}^n(A)$  is endowed with the topology induced by the valuation on  $A$ . Hence, we simply regard  $\mathbb{P}^n(A)$  as a set of points equipped with this topology. For more details, see the remark below. We refer the reader to [2] for the rigid geometry and the following fact.

**Lemma 2.1.** *The specialization map  $\mathrm{Sp}_A : \mathbb{P}^n(A) \rightarrow \mathbb{P}^n(\mathbf{k})$  is continuous and surjective.*

We begin to pin down the suitable topology for formulating Bertini-type theorems in mixed characteristic. Let  $(R, \mathfrak{m}, \mathbf{k})$  be a local Noetherian ring. Then we say that a reduced local ring  $(R, \mathfrak{m}, \mathbf{k})$  is of *mixed characteristic*  $p > 0$ , if every component of the total ring of fractions of  $R$  is of characteristic zero and the residue field  $\mathbf{k}$  is of characteristic  $p$ .

Now we assume that  $(A, \pi_A, \mathbf{k})$  is a complete discrete valuation ring such that  $\pi_A A = pA$  and there is an injection  $A \hookrightarrow R$  of rings, which induces an isomorphism on residue fields, say  $\mathbf{k} = A/\pi_A A \simeq R/\mathfrak{m}$ . Such  $A$  is called a *coefficient ring* of  $R$ .

*Example 2.2.* Let

$$R := \mathbb{Z}_p[[x, y]]/(p - xy).$$

Then  $R$  is a finite extension of  $\mathbb{Z}_p[[x + y]]$  by the Eisenstein equation  $t^2 - (x + y)t + p = 0$  and  $\mathbb{Z}_p$  is a coefficient ring of  $R$ .

In what follows, a coefficient ring will be considered as a fixed one. We denote by  $\mathbf{Loc.alg}_A$  the category of local  $A$ -algebras that are obtained as quotients of all local rings having  $A$  as their fixed coefficient ring. Note that objects of  $\mathbf{Loc.alg}_A$  also include local  $\mathbf{k}$ -algebras.

**Definition 2.3.** With the notation as above, assume that  $\mathbf{P}$  is a ring-theoretic property on Noetherian rings and fix  $(R, \mathfrak{m}, \mathbf{k}) \in \mathbf{Loc.alg}_A$ . We say that the “local Bertini theorem” holds for  $\mathbf{P}$ , if a set of minimal generators  $x_0, \dots, x_n$  of  $\mathfrak{m}$  is given, then there exists a Zariski dense open subset  $U \subseteq \mathbb{P}^n(\mathbf{k})$  such that  $R/\mathfrak{x}_a R$  has  $\mathbf{P}$  for all  $a = (a_0, \dots, a_n) \in \mathrm{Sp}_A^{-1}(U) \subseteq \mathbb{P}^n(A)$ , where we put

$$\mathfrak{x}_a = \sum_{i=0}^n a_i x_i.$$

The “local Bertini theorem” can be formulated in a different way. For example, the maximal ideal  $\mathfrak{m}$  may be replaced with a smaller ideal. But we adopt the above definition.

*Remark 2.4.* The naturality of the above formalism is explained as follows. We endow  $\mathbb{P}^n(\mathbf{k})$  with the Zariski topology. Let  $\bar{f} \in \mathbf{k}[x_0, \dots, x_n]$  be a nonzero homogeneous polynomial, let  $f \in A[x_0, \dots, x_n]$  be any fixed homogeneous lifting of  $\bar{f}$ , and let  $U_{\bar{f}} \subseteq \mathbb{P}^n(\mathbf{k})$  be an open subset defined by  $\bar{f} \neq 0$ . Then the inverse image of the open subset  $U_{\bar{f}}$  under the map  $\mathrm{Sp}_A : \mathbb{P}^n(A) \rightarrow \mathbb{P}^n(\mathbf{k})$  can be described as follows. We have

$$\mathrm{Sp}_A^{-1}(U_{\bar{f}}) = \{a = (a_0 : \dots : a_n) \in \mathbb{P}^n(A) \mid |f(a)| = 1\}$$

for the normalized valuation  $|\cdot| : A \rightarrow \mathbb{R}_{\geq 0}$ , which is an admissible open subset of  $\mathbb{P}^n(A)$ . Our objective is to show that this topology is adequate in formulating the local Bertini theorem in the mixed characteristic case.

The following proposition is indispensable for the proof of Theorem 4.3 and Theorem 4.4. However, the proposition fails for a finite field.

**Proposition 2.5** ([12]; Proposition 3.3). *Let  $U \subseteq \mathbb{A}^n(L)$  be any non-empty Zariski open subset for an infinite field  $L$ . Then  $U$  is dense. Furthermore, if  $K \rightarrow L$  is any field extension such that  $K$  is infinite, the intersection  $U \cap \mathbb{A}^n(K)$  is also a Zariski dense open subset. The above assertions hold over the projective space as well.*

### 3. DISCUSSION ON BASIC ELEMENTS

Let  $(R, \mathfrak{m})$  be a local Noetherian ring and let  $M$  be a finitely generated  $R$ -module. The depth of  $M$ , denoted by  $\text{depth}_R M$ , is the maximal length of  $M$ -regular sequences. For  $\mathfrak{p} \in \text{Spec } R$ ,  $\mu_{\mathfrak{p}}(M)$  stands for the number of minimal generators of the  $R_{\mathfrak{p}}$ -module  $M_{\mathfrak{p}}$ . The following notion is due to Swan.

**Definition 3.1** (Swan). Let  $M$  be a module over a ring  $A$  and let  $\mathfrak{p}$  be its prime ideal. An element  $m \in M$  is called *basic* at  $\mathfrak{p}$ , if  $\mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M/A \cdot m) = 1$ . More generally, a set of elements  $m_1, \dots, m_n$  of  $M$  is called *k-fold basic* at  $\mathfrak{p}$ , if  $\mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M/\sum_{i=1}^n A \cdot m_i) \geq k$ ; that is,  $N = A \cdot m_1 + \dots + A \cdot m_n$  contains at least  $k$  minimal generators at  $\mathfrak{p}$ .

Let  $M^{(r)} := M/\sum_{i=1}^r A \cdot m_i$  for a set of elements  $m_1, \dots, m_k$  of  $M$  and  $r$  satisfying  $0 \leq r \leq k-1$  and pick a prime ideal  $\mathfrak{p}$  of  $A$ . Then

$$\mu_{\mathfrak{p}}(M^{(r)}) - \mu_{\mathfrak{p}}(M^{(r)}/A \cdot m_{r+1}) = 1 \iff m_{r+1} \notin \mathfrak{p}M_{\mathfrak{p}}^{(r)}$$

for  $0 \leq r \leq k-1$  by Nakayama's lemma. In other words,  $m_1, \dots, m_k$  form partial generators of the  $k(\mathfrak{p})$ -vector space  $M \otimes_A k(\mathfrak{p})$ .

We shall use (finite) Kähler differentials. For a complete local ring  $(R, \mathfrak{m})$  with its coefficient ring  $A$ , the usual module of Kähler differentials  $\Omega_{R/A}$  is not a finite  $R$ -module. Instead, one uses the completed module  $\widehat{\Omega}_{R/A}$ . This is the  $\mathfrak{m}$ -adic completion of  $\Omega_{R/A}$  and it is a finite  $R$ -module. It can be also defined as follows. Let  $I$  denote the kernel of the map  $\mu : R \widehat{\otimes}_A R \rightarrow R$  defined by  $\mu(a \otimes b) = ab$ . Then  $\widehat{\Omega}_{R/A} := I/I^2$ . The connection of Kähler differential modules with the symbolic power ideals is expressed by the following fact (see [1], Lemma 2.2 for its proof. As a remark, there is an isomorphism  $\text{Der}_{\mathbb{Z}}(R_{\mathfrak{p}}, M_{\mathfrak{p}}) \simeq \text{Der}_{\mathbb{Z}}(R, M_{\mathfrak{p}})$ ).

**Lemma 3.2.** *Let  $M$  be a module over a ring  $R$ , let  $\mathfrak{p}$  be a prime of  $R$ , and let  $d : R \rightarrow M$  be a derivation. If for  $x \in R$ ,  $dx \in M$  is basic at  $\mathfrak{p}$ , then  $x \notin \mathfrak{p}^{(2)}$ .*

For a ring  $A$  and an open subset  $U \subseteq \text{Spec } A$ , we set

$$\dim_U(R/I) := \dim(V(I) \cap U)$$

for an ideal  $I$  of  $A$ . The authors are grateful to Prof. V. Trivedi for explaining the proof of following lemma, which is taken from ([1], Lemma 1.2). We give its proof for the readers. As an important remark, if  $M$  is a finite projective  $R$ -module, an element  $m \in M$  is basic at all primes of  $R$  if and only if  $R \cdot m \subseteq M$  is a direct summand. The property that  $m \in M$  is basic at  $\mathfrak{p} \in \text{Spec } A$  is stable under taking quotient and localization of  $A$ . So, when applying Noetherian induction, one may assume that the ring is reduced. Then one can use the fact that a finite module over a reduced ring is generically free and a submodule of a module is generically a direct summand. We denote by  $\text{Min}_R(I)$  the set of all minimal prime divisors of an ideal  $I \subseteq R$ .

**Lemma 3.3** (Flenner). *Suppose that  $R$  is a Noetherian ring,  $M$  is a finite  $R$ -module,  $U \subseteq \text{Spec } R$  is a Zariski open subset, and  $\{m_1, \dots, m_n\}$  is a set of elements of  $M$ , which generates the submodule  $N \subseteq M$ . Suppose that we have  $t \in \mathbb{Z}$  (which can be negative) such that*

$$\mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M/N) \geq \dim_U(R/\mathfrak{p}) - t$$

*for every  $\mathfrak{p} \in U$ . Let  $U'$  be the inverse image of  $U$  under the map  $\text{Spec } R[X_1, \dots, X_n] \rightarrow \text{Spec } R$ . Then there exists an ideal  $(F_1, \dots, F_r) \subseteq R[X_1, \dots, X_n]$  such that*

$$\dim(U' \cap V(F_1, \dots, F_r)) \leq n + t$$

*and the element*

$$\sum_{i=1}^n m_i \otimes X_i \in M \otimes_R R[X_1, \dots, X_n]$$

*is basic on  $D(F_1, \dots, F_r) \cap U'$ .*

*Proof.* We may assume that all rings that appear below are reduced by taking their reduced part and we are still in the stated hypothesis. If  $U = \emptyset$ , then there is nothing to prove. So assume  $U \neq \emptyset$  and further  $R = R_{\text{red}}$ . Then by the generic freeness, there exists  $f \in A$  such that the following conditions hold:

- (1)  $M[f^{-1}]$  is  $R[f^{-1}]$ -free;
- (2)  $N[f^{-1}] = \bigoplus_{i=1}^r R[f^{-1}] \cdot m_i$  for some  $r \leq n$ , after reordering  $\{m_1, \dots, m_n\}$  if necessary;
- (3)  $N[f^{-1}]$  is an  $R[f^{-1}]$ -direct summand of  $M[f^{-1}]$ ;
- (4)  $r \geq \mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M/N)$  for every  $\mathfrak{p} \in \text{Min}_R((0)) \cap U$ .

Let us consider the lemma over  $U \cup \text{Spec } R[f^{-1}]$ . Then we may find  $\beta_{k,s} \in R[f^{-1}]$  for which  $m_s = \sum_{k=1}^r \beta_{k,s} m_k$  for all  $s > r$ . So we have

$$\sum_{k=1}^n m_k \otimes X_k = \sum_{k=1}^r m_k \otimes (X_k + \sum_{s=r+1}^n \beta_{k,s} X_s) \in M[f^{-1}] \otimes_{R[f^{-1}]} R[f^{-1}][X_1, \dots, X_n]$$

and let  $F_k := X_k + \sum_{s=r+1}^n \beta_{k,s} X_s$ ,  $1 \leq k \leq r$ . From this presentation, it is easy to see that the element

$$\sum_{k=1}^n m_k \otimes X_k$$

is basic on  $D(F_1, \dots, F_r) \subseteq \text{Spec } R[f^{-1}][X_1, \dots, X_n]$ . Let

$$\pi : \text{Spec } R[f^{-1}][X_1, \dots, X_n] \rightarrow \text{Spec } R[X_1, \dots, X_n],$$

be the natural mapping and let  $V(F_1, \dots, F_r)^{\text{cl}}$  be the Zariski closure of the image of  $V(F_1, \dots, F_r) \subseteq \text{Spec } R[f^{-1}][X_1, \dots, X_n]$  under  $\pi$ . Under the condition (4) above, we claim the following inequality:  $r \geq \dim_U R - t$ . Indeed, there exists  $\mathfrak{p} \in \text{Min}_R((0)) \cap U$  such that  $\dim_U(R) = \dim_U(R/\mathfrak{p})$  ( $R$  may not be equi-dimensional), and thus we have

$$r \geq \mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M/N) \geq \dim_U(R/\mathfrak{p}) - t = \dim_U(R) - t.$$

Then it follows from the equality  $\text{ht}(F_1, \dots, F_r) = r$  that

$$\dim(U' \cap V(F_1, \dots, F_r)^{\text{cl}}) \leq \dim_U(R) + n - r \leq n + t,$$

as desired. Note that even when  $r = 0$ , the above proof still works and  $(F_1, \dots, F_r) = 0$  in this case. We have thus proved the lemma over  $U \cap \text{Spec } R[f^{-1}]$ .

Next, suppose we have proved the lemma over  $U \cap \text{Spec}(R/I)$  for any nonzero ideal  $I \subseteq R$ . In particular, the lemma holds on both  $U \cap \text{Spec}(R/(f))$  and  $U \cap \text{Spec}(R[f^{-1}])$  for  $f \in R$  as above, respectively. By combining these together, we obtain an ideal  $(F_1, \dots, F_r)$  together with the required dimension bound (for this, one uses the fact  $V(I \cdot J) = V(I) \cup V(J)$  for ideals  $I, J$ ). Thus, it suffices to prove the lemma over  $U \cap \text{Spec}(R/(f))$ . Suppose we have constructed an increasing chain of radical ideals:

$$\sqrt{(f)} := \sqrt{(f_1)} \subseteq \sqrt{(f_1, f_2)} \subseteq \dots \subseteq \sqrt{(f_1, \dots, f_i)} \subseteq \dots$$

and  $\bar{f}_{i+1} \in R_i$  is chosen, so that the conditions (1) through (4) above hold on  $R_i$  and the conclusion of the lemma on  $R_i[\bar{f}_{i+1}^{-1}]$  with  $R_i := R/\sqrt{(f_1, \dots, f_i)}$ . As previously, if

$$U \cap \text{Spec } R_i = \emptyset,$$

then we could stop the induction at  $R_{i-1}$ . Since  $R$  is Noetherian, the above sequence stabilizes for some  $i$ . Then this implies that  $\bar{f}_{i+1} = 0$ . If  $R_i \neq 0$ , then we could have found a nonzero  $\bar{f}_{i+1}$  due to the generic freeness. So,  $\bar{f}_{i+1} = 0$  implies that we have chosen  $\bar{f}_i$  as a unit in  $R_{i-1}$ , making  $R_i = 0$ . Therefore, all the conditions (1) through (4) are already satisfied on  $R_{i-1}$  without localizing it, and we have already finished the proof of the case under such conditions. Therefore, an inductive argument yields a proof on  $U \cap \text{Spec } R$ . This completes the proof of the lemma.  $\square$

As a corollary, we obtain the following.

**Corollary 3.4.** *Let  $R$  be a Noetherian ring and suppose  $M$  is a finite  $R$ -module. Fix  $x \in M$  and define  $U_x$  as the subset of  $\text{Spec } R$  such that  $x \in M$  is basic at every point of  $U_x$ . Then  $U_x$  is a (possibly empty) constructible subset of  $\text{Spec } R$ .*

*Proof.* Recall that for a finite projective  $R$ -module  $N$ ,  $x \in N$  is basic at  $\mathfrak{p} \in \text{Spec } R$  if and only if  $R_{\mathfrak{p}} \cdot x$  spans a direct summand of  $N_{\mathfrak{p}}$ , which is an open property. We may assume that  $R$  is reduced, so that there exists  $f \in R$  for which  $M[f^{-1}]$  is a free  $R[f^{-1}]$ -module. Hence there exists an open subset  $V \subseteq \text{Spec } R[f^{-1}]$  such that  $x \in M$  is basic at  $\mathfrak{p} \in \text{Spec } R \iff \mathfrak{p} \in V$ . By Noetherian induction applied to  $V(f)$ , we find a maximal constructible subset  $Z \subseteq V(f)$  on which  $x \in M$  is basic. So  $U_x := V \cup Z$  is the sought one.  $\square$

#### 4. MAIN THEOREMS

In this section, we establish main theorems. Let us collect some facts for later use. A local domain  $S$  is *catenary*, if and only if  $\text{ht } \mathfrak{p} + \dim S/\mathfrak{p} = \dim S$  for all  $\mathfrak{p} \in \text{Spec } S$ . Let  $(R, \mathfrak{m})$  be an equi-dimensional catenary local ring and suppose  $x \in \mathfrak{m}$  is a parameter. Then  $\dim R[x^{-1}] = \dim R - 1$ . Indeed, any prime ideal  $P$  of  $R$  that is the largest possible one such that  $x \notin P$  has  $\text{ht } P = \dim R - 1$ . In particular, let  $P$  and  $P'$  be such prime ideals. Then  $\text{ht } P = \text{ht } P'$  and this fact plays a role later. We also need some lemmas:

**Lemma 4.1.** *Let  $(A, \pi_A, \mathbf{k})$  be a discrete valuation ring and let  $f \in A[y_1, \dots, y_d]$  be a nonzero (possibly constant) polynomial. Then there exists  $t \in \mathbb{Z}_{\geq 0}$  such that  $\pi_A^{-t} f \in A[y_1, \dots, y_d]$  and the reduction of  $\pi_A^{-t} f$  modulo  $\pi_A$  is a nonzero (possibly constant) polynomial in  $\mathbf{k}[y_1, \dots, y_d]$ .*

*Proof.* The proof goes by induction on  $d$ . If  $d = 1$ , we write  $f = a_m y_1^m + a_{m-1} y_1^{m-1} + \dots + a_0$  with  $a_i \in A$ . Let  $0 \leq h \leq m$  be such that the valuation  $v(a_h)$  is the smallest in the set  $\{v(a_i) \mid 0 \leq i \leq m\}$  and write  $a_h = (\text{unit}) \cdot \pi_A^t$ . Dividing  $f$  by  $\pi_A^t$ , we get the desired polynomial.

In general, write  $f = b_n y_d^n + b_{n-1} y_d^{n-1} + \dots + b_0$  for  $b_i \in A[y_1, \dots, y_{d-1}]$ . Applying the induction hypothesis to every  $b_i$ , we may find  $t_i \in \mathbb{Z}_{\geq 0}$  such that  $\pi_A^{-t_i} b_i$  has the desired property. Let  $t_s := \min\{t_i \mid 0 \leq i \leq n\}$ . Then the term  $\pi_A^{-t_s} b_s y_d^s$  modulo  $\pi_A$  is nonzero and it is clear that  $\pi_A^{-t_s} f$  is contained in  $A[y_1, \dots, y_d]$ . Hence it suffices to put  $t := t_s$ .  $\square$

**Lemma 4.2.** *Let  $(R, \mathfrak{m})$  be a universally catenary local domain and consider the polynomial algebra  $R[\underline{X}]$ . Let  $T$  be the localization of  $R[\underline{X}]$  at the prime ideal  $\mathfrak{m}R[\underline{X}]$ . Then the induced map  $\text{Spec } T \rightarrow \text{Spec } R$  is bijective.*

*Proof.* The map  $R \rightarrow T$  is local flat and  $\dim R = \dim T$  and thus,  $\text{Spec } T \rightarrow \text{Spec } R$  is surjective. To show injectivity, note that  $\mathfrak{p}T \in \text{Spec } T$  for any  $\mathfrak{p} \in \text{Spec } R$  by the construction of  $T$  from  $R$ . Now take  $P \in \text{Spec } T$  such that  $P \cap R = \mathfrak{p}$ . Then we have  $\mathfrak{p}T \subseteq P$  and thus,  $\text{ht}(\mathfrak{p}T) \leq \text{ht } P$ . Then we need to show that  $\mathfrak{p}T = P$ . For a contradiction, assume  $\mathfrak{p}T \subsetneq P$ .



First, consider the case that  $\text{ht}(\mathfrak{p}T) = \text{ht } P$ . Then this is a contradiction to  $\text{ht}(\mathfrak{p}T) < \text{ht } P$ . Hence  $\mathfrak{p}T = P$  in this case.

Next, consider the case that  $\text{ht}(\mathfrak{p}T) < \text{ht } P$  and take a saturated chain of primes in  $T$ :

$$\mathbf{P}_1 : 0 = P_0 \subsetneq \cdots \subsetneq \mathfrak{p}T \subsetneq \cdots \subsetneq P \subsetneq \cdots \subsetneq \mathfrak{m}_T,$$

where  $\mathfrak{m}_T$  is the unique maximal ideal of  $T$ . Then by casting away the redundant primes (if they exist),

$$\mathbf{P}_2 : 0 = P_0 \cap R \subsetneq \cdots \subsetneq \mathfrak{p} = P \cap R \subsetneq \cdots \subsetneq \mathfrak{m}$$

is a saturated chain of primes in  $R$ , since  $\mathfrak{q}T$  is a prime ideal and  $\mathfrak{q} = \mathfrak{q}T \cap R$  for any  $\mathfrak{q} \in \text{Spec } R$ . However, the assumption  $\text{ht}(\mathfrak{p}T) < \text{ht } P$  shows that the length of  $\mathbf{P}_2$  is strictly shorter than that of  $\mathbf{P}_1$ . Since both  $R$  and  $T$  are catenary local domains and  $\dim R = \dim T$ , this gives a contradiction and we must have  $\text{ht}(\mathfrak{p}T) = \text{ht } P$ . Hence  $\mathfrak{p}T = P$  in any case and  $\text{Spec } T \rightarrow \text{Spec } R$  is injective, as desired.  $\square$

The authors do not know if the above lemma holds in a more general setting for a local ring  $(R, \mathfrak{m})$ .

**Theorem 4.3.** *Let  $(R, \mathfrak{m}, \mathbf{k})$  be a complete local domain of mixed characteristic  $p > 0$  and suppose the following conditions:*

- (1) *let  $A \rightarrow R$  be a coefficient ring map for a complete discrete valuation ring  $(A, \pi_A)$ ;*
- (2) *let  $x_0, x_1, \dots, x_d$  be a fixed set of minimal generators of  $\mathfrak{m}$ ;*
- (3) *the residue field of  $R$  is infinite.*

*Then there exists a Zariski dense open subset  $V \subseteq \mathbb{P}^d(\mathbf{k})$  such that we have*

$$\mathbf{x}_a = \sum_{i=0}^d a_i x_i \notin \mathfrak{p}^{(2)}$$

*for every prime  $\mathfrak{p}$  of  $R$  and  $a = (a_0 : \cdots : a_d) \in \text{Sp}_A^{-1}(V)$ .*

It is noted that  $x_0, x_1, \dots, x_d$  are not a system of parameters of  $R$  unless it is regular.

*Proof.* It is clear that for  $a \in \mathbb{A}^{d+1}(A)$ ,  $\mathfrak{p} \in \text{Spec } R$ , and  $u \in A^\times$ ,

$$\sum_{i=0}^d a_i x_i \notin \mathfrak{p}^{(2)} \iff u \left( \sum_{i=0}^d a_i x_i \right) \notin \mathfrak{p}^{(2)}.$$

This implies that our statement for  $x_a$  holds on  $\mathbb{P}^d(A)$ . Note that the  $R$ -module  $\widehat{\Omega}_{R/A}$  is generated by  $dx_0, \dots, dx_d$  and  $\text{Supp } \widehat{\Omega}_{R/A} = \text{Spec } R$ . We first establish the theorem on the open subset:

$$D(\mathfrak{m}) = D(x_0) \cup \cdots \cup D(x_d).$$

Let us consider the proof over the ring  $R[x_i^{-1}]$ . It follows from ([1], Lemma 2.6, or [13], Lemma 2) that for  $\mathfrak{p} \in D(x_i)$ , we have by the preceding remark on the dimension of  $R[x_i^{-1}]$  that

$$\mu_{\mathfrak{p}}(\widehat{\Omega}_{R/A}) \geq \dim(R/\mathfrak{p}) - 1 = \dim_{D(x_i)}(R/\mathfrak{p}).$$

Under the notation being as above and applying Lemma 3.3 for the  $R$ -module  $\widehat{\Omega}_{R/A}$ , there is an ideal  $(F_1, \dots, F_r) \subseteq R[X_0, \dots, X_d]$  such that

$$\dim_{D(x_i)'} R[X_0, \dots, X_d]/(F_1, \dots, F_r) \leq d + 1 \cdots (*),$$

where  $D(x_i)'$  is the inverse image of  $D(x_i)$  under the map  $\text{Spec } R[X_0, \dots, X_d] \rightarrow \text{Spec } R$ , and the element

$$\sum_{k=0}^d dx_k \otimes X_k \in \widehat{\Omega}_{R/A} \otimes_R R[x_i^{-1}][X_0, \dots, X_d]$$

is basic on  $D(F_1, \dots, F_r) \subseteq \text{Spec } R[x_i^{-1}][X_0, \dots, X_d]$ . Let  $T$  denote the localization of  $R[X_0, \dots, X_d]$  at the prime ideal  $\mathfrak{m}R[X_0, \dots, X_d]$ . Then we have a flat local map of local rings:

$$(R, \mathfrak{m}) \rightarrow (T, \mathfrak{m}_T)$$

in which all rings are catenary local domains, and  $\dim R = \dim T$ .

Now assume that  $(F_1, \dots, F_r)T[x_i^{-1}]$  is a proper ideal and let  $P \subseteq T[x_i^{-1}]$  be any minimal prime divisor of  $(F_1, \dots, F_r)T[x_i^{-1}]$ . Then combining the dimension equality for the catenary local domain  $T$ , together with the inequality  $(*)$  above, it follows that

$$\text{ht } P = \dim T[x_i^{-1}] = \dim T - 1.$$

Thus,  $Z_i := V(F_1, \dots, F_r) \cap \text{Spec } T[x_i^{-1}]$  consists of finitely many maximal ideals, and

$$\sum_{k=0}^d dx_k \otimes X_k$$

is basic on  $\text{Spec } T[x_i^{-1}] \setminus Z_i$ . Since the induced map  $\phi : \text{Spec } T[x_i^{-1}] \rightarrow \text{Spec } R[x_i^{-1}]$  is bijective by Lemma 4.2, we have  $\phi^{-1}(\phi(Z_i)) = Z_i$  and  $\phi(Z_i)$  is a finite closed subset, which implies the following: One can find  $h_1, \dots, h_m \in R$  such that  $Z_i = V(h_1, \dots, h_m) \subseteq \text{Spec } T[x_i^{-1}]$ .

Since an open set is covered by basic open subsets, Noetherian induction reduces to proving the theorem on  $D(g \cdot x_i) \subseteq \text{Spec } T$  for some  $g \in R$  such that  $\widehat{\Omega}_{R/A} \otimes_R T[(g \cdot x_i)^{-1}]$  is a  $T[(g \cdot x_i)^{-1}]$ -free module,  $\sum_{k=0}^d dx_k \otimes X_k$  spans the direct summand of  $\widehat{\Omega}_{R/A} \otimes_R T[(g \cdot x_i)^{-1}]$ . Moreover,  $D(g \cdot h_1 \cdots h_m) \cap Z_i = \emptyset$  in  $\text{Spec } T[x_i^{-1}]$  by the lemma of generic freeness. But since  $T$  is the localization of  $R[X_0, \dots, X_d]$ , there exists a polynomial  $G_i \in R[X_0, \dots, X_d] \setminus \mathfrak{m}R[X_0, \dots, X_d]$  such that

$$\sum_{k=0}^d dx_k \otimes X_k$$

is basic on  $D(g \cdot h_1 \cdots h_m \cdot G_i) \subseteq \text{Spec } R[x_i^{-1}][X_0, \dots, X_d]$ . Hence by ([1], Lemma 1.1), the element

$$\sum_{k=0}^d a_k dx_k \in M[x_i^{-1}]$$

is basic on  $D(g \cdot h_1 \cdots h_m \cdot G_i(a))$  for  $a = (a_0, \dots, a_d) \in \mathbb{A}^{d+1}(A)$ . Assume that  $G_i(a) \notin \mathfrak{m}$  (this is actually satisfied for some  $a$ , since the residue field of  $R$  (or  $A$ ) is infinite). Then

$$\sum_{k=0}^d a_k dx_k \in M[x_i^{-1}]$$

is basic on  $D(g \cdot h_1 \cdots h_m) \subseteq \text{Spec } R[x_i^{-1}]$ . Now we get the following implication. Take  $G_i(a) \notin \mathfrak{m}$ . Then for  $\mathfrak{p} \in \text{Spec } R[x_i^{-1}] \setminus \phi(Z_i)$ , it follows from Lemma 3.2 that

$$\sum_{k=0}^d a_k x_k \notin \mathfrak{p}^{(2)}.$$

By carrying out the same process for all  $D(x_0), \dots, D(x_d)$ , we find polynomials  $G_0, \dots, G_d \in R[X_0, \dots, X_d] \setminus \mathfrak{m}R[X_0, \dots, X_d]$  with the aforementioned properties. Let

$$G = \prod_{i=0}^d G_i \in R[X_0, \dots, X_d] \setminus \mathfrak{m}R[X_0, \dots, X_d].$$

Then if  $G(a) \notin \mathfrak{m}$ , there exists a finite set of primes  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$  in  $R$  such that  $\text{ht } \mathfrak{p}_i = \dim R - 1$  for all  $i$ , and for every  $\mathfrak{p} \in \text{Spec } R \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_r, \mathfrak{m}\}$ , we have

$$\mathbf{x}_a = \sum_{i=0}^d a_i x_i \notin \mathfrak{p}^{(2)}.$$

So it remains to deal with the issue on a finite set  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r, \mathfrak{m}\}$ . First, to satisfy  $x_a \notin \mathfrak{m}^2$ , it suffices to have that  $a_i \in A^\times$  for some  $i$  by Nakayama's lemma, since  $x_0, \dots, x_d$  are the minimal generators of  $\mathfrak{m}$ . So modifying  $G$ , we may assume henceforth that  $\mathbf{x}_a \notin \mathfrak{p}^2$  is satisfied for all  $\mathfrak{p} \in \text{Spec } R \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ , whenever  $G(a) \notin \mathfrak{m}$ .

Put  $\overline{V}_0 := D(\overline{G}(\overline{x})) \subseteq \mathbb{A}^{d+1}(\mathbf{k}) \setminus \{0\}$  (clearly, the origin of  $\mathbb{A}^{d+1}(\mathbf{k})$  is excluded). Then define an open subset  $V_0 \subseteq \mathbb{P}^d(\mathbf{k})$  as the image of  $\overline{V}_0$  under the geometric quotient:  $\mathbb{A}^{d+1}(\mathbf{k}) \setminus \{0\} \rightarrow \mathbb{P}^d(\mathbf{k})$ . To deal with the issue on  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ , take the homogeneous polynomial:

$$F(X_0, \dots, X_d) := \sum_{i=0}^d x_i X_i \in R[X_0, \dots, X_d].$$

To prove the theorem, it suffices to force  $\sum_{i=0}^d a_i x_i \notin \mathfrak{p}_j$  for all  $1 \leq j \leq r$ . Then for each  $1 \leq j \leq r$ , we have

$$\mathbf{x}_a = \sum_{i=0}^d a_i x_i \notin \mathfrak{p}_j \iff \overline{F}(\overline{a}_0, \dots, \overline{a}_d) \neq 0 \text{ in } R/\mathfrak{p}_j \cdots (**),$$

which is stable under taking multiplication by elements of  $A^\times$ .

Our final goal is to identify the set of points of  $\mathbb{P}^d(A)$  satisfying the condition  $(**)$  and describe it as the inverse image of an open subset under the map  $\mathrm{Sp}_A : \mathbb{P}^d(A) \rightarrow \mathbb{P}^d(\mathbf{k})$ . It is clear that the ideal of  $R$  generated by the set:  $\{F(a_0, \dots, a_d) \mid (a_0, \dots, a_d) \in \mathbb{P}^d(A)\}$  is the maximal ideal  $\mathfrak{m}$ . Since the union of all primes  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  is strictly contained in  $\mathfrak{m}$ , there exists  $(a_0, \dots, a_d) \in \mathbb{P}^d(A)$  for which  $F(a_0, \dots, a_d) \notin \mathfrak{p}_i$  for all  $i$ . We will use this fact just below.

**Case1:** Assume that  $\pi_A \in \mathfrak{p}_j$  for some  $j$ . Let  $R_j$  be the localization of  $R$  at  $\mathfrak{p}_j$ . Then  $A \rightarrow R_j$  is a flat local map of local rings, say  $R_j$  is of mixed characteristic. Let  $\mathfrak{m}_j$  be the maximal ideal of  $R_j$  and let  $\mathbf{k}_j := R_j/\mathfrak{m}_j$ . Then we have a mapping:

$$\mathbb{P}^d(\mathbf{k}) \rightarrow \mathbb{P}^d(\mathbf{k}_j).$$

The condition that the reduced polynomial  $\bar{F}(\bar{a}_0, \dots, \bar{a}_d)$  is nonzero on the projective space  $\mathbb{P}^d(\mathbf{k}_j)$  defines an open subset  $U_j \subseteq \mathbb{P}^d(\mathbf{k}_j)$ . By Proposition 2.5,  $V_j := U_j \cap \mathbb{P}^d(\mathbf{k})$  is a dense open subset, as the field  $\mathbf{k}$  is infinite.

**Case2:** Assume that  $\pi_A \notin \mathfrak{p}_j$  for some  $j$ . In this case, notations being as in (i), we have a mapping:

$$\mathbb{P}^d(\mathrm{Frac}(A)) \rightarrow \mathbb{P}^d(\mathbf{k}_j).$$

The condition  $\bar{F}(\bar{a}_0, \dots, \bar{a}_d) \neq 0$  on the projective space  $\mathbb{P}^d(\mathbf{k}_j)$  defines a Zariski open subset  $U_j \subseteq \mathbb{P}^d(\mathbf{k}_j)$ .

Proposition 2.5 implies that  $U_j \cap \mathbb{P}^d(\mathrm{Frac}(A))$  is a dense open subset which is covered by basic open subsets. Let  $U_f$  be one of those basic open subsets for a homogeneous polynomial  $f$ , where  $U_f$  has the usual meaning. We may assume that  $f \in A[X_0, \dots, X_d]$ . Furthermore, by applying Lemma 4.1 to  $f$ , it can be normalized into  $g$  so that  $g \in A[X_0, \dots, X_d]$  and  $0 \neq \bar{g} \in \mathbf{k}[X_0, \dots, X_d]$ . Consider the subset:

$$U_{|g|=1} := \{a = (a_0 : \dots : a_d) \in \mathbb{P}^d(A) \mid |g(a)| = 1\} \subseteq \mathbb{P}^d(A)$$

for the normalized valuation  $|\cdot| : A \rightarrow \mathbb{R}_{\geq 0}$ . Then we see that  $F(a_0, \dots, a_d) \notin \mathfrak{p}_j$  for all  $(a_0 : \dots : a_d) \in U_{|g|=1}$  and  $U_{|g|=1}$  is non-empty, since  $|g(a)| = 1$  if and only if  $\bar{g}(\mathrm{Sp}_A(a)) \neq 0$ . Moreover, this implies that  $\mathrm{Sp}_A^{-1}(U_{\bar{g}}) = U_{|g|=1}$ . By taking the union of all affine pieces  $U_{\bar{g}}$  coming from a basic open covering of  $U_j \cap \mathbb{P}^d(\mathrm{Frac}(A))$ , we obtain a desired open subset  $V_j \subseteq \mathbb{P}^d(\mathbf{k})$ , as in **Case1**.

Combining both **Case1** and **Case2** together,  $V_1 \cap \dots \cap V_r$  is a non-empty open subset of  $\mathbb{P}^d(\mathbf{k})$ . Hence  $V$  is defined as the intersection  $V_0 \cap V_1 \cap \dots \cap V_r$ , where  $V_0$  is defined by  $G(x)$  as previously. This completes the proof of the theorem.  $\square$

The techniques used in the proof of the above theorem will play an important role in the next main theorem.

**Theorem 4.4** (Local Bertini Theorem). *Let  $(R, \mathfrak{m}, \mathbf{k})$  be a complete local domain of mixed characteristic  $p > 0$  and suppose the following conditions:*

- (1) *let  $A \rightarrow R$  be a coefficient ring map for a complete discrete valuation ring  $(A, \pi_A)$ ;*
- (2) *let  $x_0, x_1, \dots, x_d$  be a fixed set of minimal generators of  $\mathfrak{m}$ ;*
- (3)  *$R$  is normal, of depth  $R \geq 3$ , and the residue field  $\mathbf{k}$  is infinite.*

*Then there exists a Zariski dense open subset  $U \subseteq \mathbb{P}^d(\mathbf{k})$  satisfying the following properties. For any  $a = (a_0 : \dots : a_d) \in \mathrm{Sp}_A^{-1}(U)$ , the quotient  $R/\mathbf{x}_a R$  is a normal domain of mixed characteristic  $p > 0$ , where we put*

$$\mathbf{x}_a := \sum_{i=0}^d a_i x_i.$$

*Proof.* The first step of the proof of the theorem has been completed in Theorem 4.3. Taking  $V \subseteq \mathbb{P}^d(\mathbf{k})$  as given in Theorem 4.3 and denoting by  $\mathrm{Reg}(R)$  the regular locus of  $R$ , we find that

$$\mathrm{Reg}(R) \cap V(\mathbf{x}_a) \subseteq \mathrm{Reg}(R/\mathbf{x}_a R) \cdots (1)$$

for all  $a = (a_0, \dots, a_d) \in \mathrm{Sp}_A^{-1}(V)$ . Let us explain some basic ideas. In this proof, after some preparations, we give a candidate of  $U \subseteq \mathbb{P}^d(\mathbf{k})$  for our theorem as a Zariski open subset  $U \subseteq V$ . In **Step1** below we prove that  $U$  is non-empty. Then in **Step2**, we will show that the quotients  $R/\mathbf{x}_a R$  with  $a \in \mathrm{Sp}_A^{-1}(U)$  satisfy the well-known conditions  $(R_1)$  and  $(S_2)$  and thus are normal. Finally in **Step3**, we will show that the quotients  $R/\mathbf{x}_a R$  with  $a \in \mathrm{Sp}_A^{-1}(U)$  are of mixed characteristic.

**Step1:** Let  $X := \mathrm{Spec} R - V(\mathfrak{m})$ . Then since  $R$  is a complete local domain, the singular locus  $\mathrm{Sing}(X)$  is a proper closed subset. Hence the set of minimal primes in  $\mathrm{Sing}(X)$  is finite, and let

$$Q_1 := \{\mathfrak{p} \in X \mid \mathfrak{p} \text{ is a minimal prime in } \mathrm{Sing}(X)\}.$$

Note that every prime in  $Q_1$  has height  $\geq 2$  (due to the  $(S_2)$  condition on  $R$ ). On the other hand,

$$Q_2 := \{\mathfrak{p} \in X \mid \mathrm{depth} R_{\mathfrak{p}} = 2 \text{ and } \dim R_{\mathfrak{p}} > 2\}$$

is also a finite set ([1], Lemma 3.2 and the  $(S_2)$  condition on  $R$ ). Now let  $Q_1 \cup Q_2 := \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$  and let

$$F(X_0, \dots, X_d) := \sum_{i=0}^d x_i X_i \in R[X_0, \dots, X_d].$$

Then for each  $1 \leq j \leq m$ , it follows that

$$\mathbf{x}_a = \sum_{i=0}^d a_i x_i \notin \mathfrak{p}_j \iff \overline{F}(\overline{a}_0, \dots, \overline{a}_d) \neq 0 \text{ in } R/\mathfrak{p}_j.$$

Then as argued in **Case1** and **Case2** in the proof of Theorem 4.3, we obtain  $Z_1, \dots, Z_m$ , which are non-empty open subsets of  $\mathbb{P}^d(\mathbf{k})$ . Put  $Z := Z_1 \cap \dots \cap Z_m \cdots (2)$ .

Next pick  $\mathfrak{p} \in X \cap V(x_a)$  such that  $\text{ht } \mathfrak{p} \geq 2$  and assume that  $\mathbf{x}_a = \sum_{i=0}^d a_i x_i$  satisfies (1) and  $\mathbf{x}_a$  avoids all primes in  $Q_1 \cup Q_2$ .

**Step2:** (i) If  $\text{ht } \mathfrak{p} > 2$ , then since  $\mathbf{x}_a$  avoids all primes in  $Q_2$ , it follows that  $\dim(R/\mathbf{x}_a R)_{\mathfrak{p}} \geq 2$  and  $\text{depth}(R/\mathbf{x}_a R)_{\mathfrak{p}} \geq 2$ .

(ii) If  $\text{ht } \mathfrak{p} = 2$ , then since  $\mathbf{x}_a$  avoids all primes ideal in  $Q_1$  and the height of every prime in  $Q_1$  is at least 2, it follows that  $R_{\mathfrak{p}}$  is regular. By (1), one finds that  $(R/\mathbf{x}_a R)_{\mathfrak{p}}$  is a discrete valuation ring. On the other hand, the hypothesis that  $\text{depth}(R) \geq 3$  implies that  $\text{depth}(R/\mathbf{x}_a R) \geq 2$ . Hence  $R/\mathbf{x}_a R$  is a normal domain in view of Serre's normality criterion.

Finally, it remains to make the quotient  $R/\mathbf{x}_a R$  into a local ring of mixed characteristic  $p > 0$ .

**Step3:** Let  $\{q_1, \dots, q_n\}$  be a set of all height-one primes of  $R$  lying above  $\pi_A$ . Then again, applying the discussion of **Case1** and **Case2** in the proof of Theorem 4.3 to  $\{q_1, \dots, q_n\}$ , we find a non-empty open subset  $W \subseteq \mathbb{P}^d(\mathbf{k}) \cdots (3)$ , which avoids the union  $q_1 \cup \dots \cup q_n$ .

Combining (1), (2), together with (3), and taking a non-empty open subset  $U := V \cap W \cap Z \subseteq \mathbb{P}^d(\mathbf{k})$ , it turns out that  $\text{Sp}_A^{-1}(U) \subseteq \mathbb{P}^d(A)$  has the required property.  $\square$

*Remark 4.5.* In particular, if  $R$  is a Cohen-Macaulay normal local domain, then  $R/\mathbf{x}_a R$  is Cohen-Macaulay and normal. One can continue this process until  $\dim R = 2$  is attained. Note that the module-finite extension of local rings  $A[[z_1, \dots, z_n]] \rightarrow R$  is flat if and only if  $R$  is Cohen-Macaulay, by the Auslander-Buchsbaum formula.

Next, let us consider the case when the residue field is finite. Let  $(R, \mathfrak{m}, \mathbb{F})$  be a complete normal local domain of mixed characteristic  $p > 0$  with finite residue field  $\mathbb{F}$ . In other words,  $R$  is a finite extension of  $W(\mathbb{F})[[z_1, \dots, z_n]]$ , where  $W(\mathbb{F})$  is the ring of Witt vectors of  $\mathbb{F}$ . Let  $\mathcal{O} := W(\mathbb{F})$  for simplicity and let  $\widehat{\mathcal{O}^{\text{ur}}}$  be the completion of the maximal unramified extension of  $\mathcal{O}$ . Then  $W(\overline{\mathbb{F}}) = \widehat{\mathcal{O}^{\text{ur}}}$ . Put

$$R_{\widehat{\mathcal{O}^{\text{ur}}}} := R \widehat{\otimes}_{\mathcal{O}} \widehat{\mathcal{O}^{\text{ur}}} \text{ (resp. } R_{\mathcal{O}^{\text{ur}}} := \text{strict henselization of } R).$$

Then  $R_{\mathcal{O}^{\text{ur}}}$  is local Noetherian, but not complete. By the main result of [3],  $R_{\widehat{\mathcal{O}^{\text{ur}}}}$  is the completion of  $R_{\mathcal{O}^{\text{ur}}}$  and a normal local domain. From algebraic number theory,  $\mathcal{O}^{\text{ur}}$  is obtained from  $\mathcal{O}$  by adjoining all  $n$ -th roots of unity for  $(n, p) = 1$ . There is a structure map  $\widehat{\mathcal{O}^{\text{ur}}} \rightarrow R_{\widehat{\mathcal{O}^{\text{ur}}}}$ . We define the multiplicative map (not additive)

$$\theta_{\widehat{\mathcal{O}^{\text{ur}}}} : \overline{\mathbb{F}} \rightarrow \widehat{\mathcal{O}^{\text{ur}}}$$

as just the Teichmüller map  $\overline{\mathbb{F}} \rightarrow W(\overline{\mathbb{F}})$ . In particular, we have  $q \circ \theta_{\widehat{\mathcal{O}}^{\text{ur}}} = \text{Id}_{\overline{\mathbb{F}}}$ , where  $q : \widehat{\mathcal{O}}^{\text{ur}} \rightarrow \overline{\mathbb{F}}$  is the residue field map. There is a set-theoretic mapping:

$$\langle \theta_{\widehat{\mathcal{O}}^{\text{ur}}} \rangle : \mathbb{P}^d(\overline{\mathbb{F}}) \rightarrow \mathbb{P}^d(\widehat{\mathcal{O}}^{\text{ur}})$$

defined by  $\langle \theta_{\widehat{\mathcal{O}}^{\text{ur}}} \rangle(a) := (\theta_{\widehat{\mathcal{O}}^{\text{ur}}}(a_0) : \cdots : \theta_{\widehat{\mathcal{O}}^{\text{ur}}}(a_d))$ . Note that  $\langle q \rangle = \text{Sp}_{\widehat{\mathcal{O}}^{\text{ur}}}$  and  $\langle \theta_{\widehat{\mathcal{O}}^{\text{ur}}} \rangle$  is well-defined, since  $\theta_{\widehat{\mathcal{O}}^{\text{ur}}}$  is multiplicative. Furthermore, since the composite  $\langle q \rangle \circ \langle \theta_{\widehat{\mathcal{O}}^{\text{ur}}} \rangle$  is an identity map,  $\langle \theta_{\widehat{\mathcal{O}}^{\text{ur}}} \rangle$  is injective.

**Corollary 4.6** (Finite Residue Field Case). *Let the hypothesis be as in Theorem 4.4 for  $(R, \mathfrak{m}, \mathbb{F})$ , except that the residue field  $\mathbb{F}$  is finite. Then there exists a non-empty open subset  $U \subseteq \mathbb{P}^d(\overline{\mathbb{F}})$  such that for  $a \in \langle \theta_{\widehat{\mathcal{O}}^{\text{ur}}} \rangle(U)$ , there is a finite étale extension  $\mathcal{O} \rightarrow \mathcal{O}'$  of discrete valuation rings such that  $\mathbf{x}_a \in R_{\mathcal{O}'} := R \otimes_{\mathcal{O}} \mathcal{O}'$  and  $R_{\mathcal{O}'} / \mathbf{x}_a R_{\mathcal{O}'}$  is a normal domain of mixed characteristic  $p > 0$ .*

The proof will be done by constructing a multiplicative map  $\tilde{\theta}_{\widehat{\mathcal{O}}^{\text{ur}}} : \overline{\mathbb{F}} \rightarrow \mathcal{O}^{\text{ur}}$ , which extends to the map  $\theta_{\widehat{\mathcal{O}}^{\text{ur}}} : \overline{\mathbb{F}} \rightarrow \widehat{\mathcal{O}}^{\text{ur}}$ .

*Proof.* We keep the notation as in Theorem 4.4. First, note that  $\langle \theta_{\widehat{\mathcal{O}}^{\text{ur}}} \rangle(U) \subseteq \mathbb{P}^d(\mathcal{O}^{\text{ur}})$  and that  $x_0, x_1, \dots, x_d$  are the minimal generators of the maximal ideal of  $R_{\widehat{\mathcal{O}}^{\text{ur}}}$ . Then the hypotheses of Theorem 4.4 are fulfilled for the complete local domain  $R_{\widehat{\mathcal{O}}^{\text{ur}}}$ . First, let us construct a multiplicative map  $\tilde{\theta}_{\widehat{\mathcal{O}}^{\text{ur}}} : \overline{\mathbb{F}} \rightarrow \mathcal{O}^{\text{ur}}$  which extends to the map  $\theta_{\widehat{\mathcal{O}}^{\text{ur}}}$ . The local ring  $\mathcal{O}^{\text{ur}}$  is constructed as the direct limit of finite unramified extensions of  $\mathcal{O}$ ;  $\mathcal{O}^{\text{ur}} = \varinjlim_{\lambda \in \Lambda} \mathcal{O}_{\lambda}$  with  $\mathcal{O}_{\lambda} = W(\mathbb{F}_{\lambda})$  and we have the Teichmüller mapping  $\mathbb{F}_{\lambda} \rightarrow \mathcal{O}_{\lambda}$  for the residue field  $\mathbb{F}_{\lambda}$  of  $\mathcal{O}_{\lambda}$ . Then we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{F}_{\lambda'} & \xrightarrow{\text{Teich}} & \mathcal{O}_{\lambda'} \\ \uparrow & & \uparrow \\ \mathbb{F}_{\lambda} & \xrightarrow{\text{Teich}} & \mathcal{O}_{\lambda} \end{array}$$

which naturally forms a direct system, so the desired map  $\tilde{\theta}_{\widehat{\mathcal{O}}^{\text{ur}}}$  is given by its direct limit. On the other hand, it is easy to see that the map  $\theta_{\widehat{\mathcal{O}}^{\text{ur}}} : \overline{\mathbb{F}} \rightarrow \widehat{\mathcal{O}}^{\text{ur}}$  factors as

$$\overline{\mathbb{F}} \xrightarrow{\tilde{\theta}_{\widehat{\mathcal{O}}^{\text{ur}}}} \mathcal{O}^{\text{ur}} \longrightarrow \widehat{\mathcal{O}}^{\text{ur}},$$

and thus we have  $\mathbf{x}_a = \sum_{i=0}^d a_i x_i \in R_{\mathcal{O}^{\text{ur}}}$  for any  $a \in \langle \theta_{\widehat{\mathcal{O}}^{\text{ur}}} \rangle(U)$ . Since the map

$$R_{\mathcal{O}^{\text{ur}}} / \mathbf{x}_a R_{\mathcal{O}^{\text{ur}}} \rightarrow R_{\widehat{\mathcal{O}}^{\text{ur}}} / \mathbf{x}_a R_{\widehat{\mathcal{O}}^{\text{ur}}}$$

is local flat,  $R_{\mathcal{O}^{\text{ur}}} / \mathbf{x}_a R_{\mathcal{O}^{\text{ur}}}$  is a normal local domain.

By what we have said above, all the coefficients of a linear form  $\mathbf{x}_a = \sum_{i=0}^d a_i x_i$  are contained in some finite subextension  $\mathcal{O} \rightarrow \mathcal{O}' \rightarrow \mathcal{O}^{\text{ur}}$ . In other words, for a finite étale extension  $R \rightarrow R_{\mathcal{O}'}$  of normal domains, the quotient  $R_{\mathcal{O}'} / \mathbf{x}_a R_{\mathcal{O}'}$  is normal.  $\square$

*Remark 4.7.* Let  $\phi : (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  be a flat local map of local rings. Then one might think of the relationship between  $R/xR$  and  $S/xS$  for a nonzero divisor  $x \in \mathfrak{m}$ . In fact, in order to use the local Bertini theorem for  $S$  in terms of  $R$ , for exmaple, assume that  $R$  and all fibers of  $\phi$  are normal. Then for any  $x$  such that  $R/xR$  is normal,  $S/xS$  is so.

It is important and necessary to answer the following question:

*Question 4.8.* Resume the hypothesis of Theorem 4.4 and assume that  $\mathbf{x}_a = u\mathbf{x}_{a'}$  for a unit  $u \in R^\times$ . Then is it true that  $u \in A^\times$ ?

This question asks that which subset of  $\mathbb{P}^d(A)$  parametrizes the set of height-one primes  $\{\mathbf{x}_a R\}$ . In other words, does it provide mutually distinct prime ideals of  $R$ ? In the next section, we will answer the above question. In fact, we need to restrict to the set of those points which are in the image of the Teichmüller mapping. This fact will be important in the proof of the control theorem, which will be discussed later. We end this section with an example, which applies Theorem 4.4 and its proof for a given normal domain  $R$ .

*Example 4.9.* This example deals with the finite residue field case, but the result is valid for any discrete valuation coefficient ring. Suppose that  $p \geq 3$  and

$$R := \mathbb{Z}_p[[x_1, x_2, x_3]] / (x_1^2 + x_2^2 + x_3^2),$$

which is a three-dimensional Cohen-Macaulay normal local domain. By Theorem 4.4, there exists an open set  $U \subseteq \mathbb{P}^3(\mathbb{F}_p)$  which does the job. Now we keep track of the following steps to find  $U$ .

- (i) Need to have  $d\mathbf{x}_a \in \widehat{\Omega}_{R/\mathbb{Z}_p}$  basic at every  $\mathfrak{p} \in \text{Spec } R$ .
- (ii) Need to determine two finite sets of primes  $Q_1$  and  $Q_2$  in Theorem 4.4.
- (iii) Need to avoid  $Q_1$  and  $Q_2$  as above.

Then we know

$$\widehat{\Omega}_{R/\mathbb{Z}_p} \simeq \frac{Rdx_1 \oplus Rdx_2 \oplus Rdx_3}{R(x_1dx_1 + x_2dx_2 + x_3dx_3)},$$

the singular locus of  $R$  is defined by the ideal  $(x_1, x_2, x_3)$ ,  $Q_1 = \{(x_1, x_2, x_3)\}$  and  $Q_2 = \emptyset$ . To get a normal ring of mixed characteristic, take  $\mathbf{x}_a := a_0p + \sum_{i=1}^3 a_i x_i$  such that

$$\bar{a} = (\bar{a}_0 : \bar{a}_1 : \bar{a}_2 : \bar{a}_3) \in U := U(z_0) \cap \left( \bigcup_{i=1}^3 U(z_i) \right) \subseteq \mathbb{P}^3(\mathbb{F}_p),$$

where  $(z_0 : \cdots : z_3)$  is the homogeneous coordinate of  $\mathbb{P}^3(\mathbb{F}_p)$ . Then  $\mathbf{x}_a \notin Q_1$ . If we assume  $a_1$  is a unit for simplicity, we see that

$$\widehat{\Omega}_{R/\mathbb{Z}_p} \left[ \frac{1}{x_3} \right] \simeq R \left[ \frac{1}{x_3} \right] dx_1 \oplus R \left[ \frac{1}{x_3} \right] dx_2$$



is a free module, in which the image of  $dx_a$  spans a direct summand. On the other hand, for  $\bar{R} := R/(x_3)$ ,

$$\hat{\Omega}_{R/\mathbb{Z}_p}/x_3 \cdot \hat{\Omega}_{R/\mathbb{Z}_p} \simeq \frac{\bar{R}dx_1 \oplus \bar{R}dx_2 \oplus \bar{R}dx_3}{\bar{R}(x_1dx_1 + x_2dx_2)}.$$

To show that the image of  $dx_a$  is basic on  $\hat{\Omega}_{R/\mathbb{Z}_p}/x_3 \cdot \hat{\Omega}_{R/\mathbb{Z}_p}$ , keep track of the same steps as above by inverting and killing first  $x_2$ , and then  $x_1$ .

Finally, if one takes  $R := \mathbb{Z}_p[[x]]$ , then  $p \notin \mathfrak{p}^{(2)}$  for every prime  $\mathfrak{p}$  of  $R$ , since  $p$  is a regular parameter. But  $dp = 0$  and so the notion of basic elements is stronger than that of non-containment in the second symbolic power of a prime ideal.

## 5. DISTINCT HYPERPLANE SECTIONS IN LOCAL BERTINI THEOREM

In this section, we give an answer to Question 4.8. Assume that  $(R, \mathfrak{m}, \mathbf{k})$  is a complete local normal domain with perfect residue field of characteristic  $p > 0$  with its coefficient ring  $W(\mathbf{k})$ , the ring of Witt vectors. Then as previously, we have the mapping:  $\langle \theta_{W(\mathbf{k})} \rangle : \mathbb{P}^d(\mathbf{k}) \rightarrow \mathbb{P}^d(W(\mathbf{k}))$ . The following proposition asserts that the parameter set of specializations in the local Bertini theorem may be identified with an open subset  $U \subseteq \mathbb{P}^d(\mathbf{k})$ . In the case  $\pi_{W(\mathbf{k})} \notin \mathfrak{m}^2$ , we put

$$\mathbf{x}_a = a_0\pi_{W(\mathbf{k})} + \sum_{i=1}^d a_i x_i.$$

**Proposition 5.1.** *Let the notation and the hypothesis be as above, and let  $U \subseteq \mathbb{P}^d(\mathbf{k})$  be a non-empty subset. Moreover, put  $x_0 = \pi_{W(\mathbf{k})}$  in the case  $\pi_{W(\mathbf{k})} \notin \mathfrak{m}^2$ . Suppose that  $\mathbf{x}_a = u\mathbf{x}_{a'}$  for  $a, a' \in \langle \theta_{W(\mathbf{k})} \rangle(U)$  and  $u \in R^\times$ . Then we have  $u \in W(\mathbf{k})^\times$ .*

*Proof.* We need to divide the proof, according to the case  $\pi_{W(\mathbf{k})} \in \mathfrak{m}^2$  or  $\pi_{W(\mathbf{k})} \notin \mathfrak{m}^2$ .

**Case1:** Assume  $\pi_{W(\mathbf{k})} \in \mathfrak{m}^2$  and denote by  $a_i$  the image of  $a_i \in W(\mathbf{k})$  under the surjection  $W(\mathbf{k})[[X_0, \dots, X_d]] \twoheadrightarrow R (X_i \mapsto x_i)$ . Let  $P$  be its kernel and let  $\mathbf{k} \rightarrow W(\mathbf{k})$  be the Teichmüller mapping. Then we prove that

$$P \subseteq \pi_{W(\mathbf{k})}W(\mathbf{k})[[X_0, \dots, X_d]] + \mathfrak{M}^2,$$

where  $\mathfrak{M} := (X_0, \dots, X_d)$ . Let  $f \in P$  be a nonzero element. Assume that

$$f \notin \pi_{W(\mathbf{k})}W(\mathbf{k})[[X_0, \dots, X_d]] + \mathfrak{M}^2$$

and derive a contradiction. For the proof of the claim, we may reduce  $W(\mathbf{k})[[X_0, \dots, X_d]]$  by  $\pi_{W(\mathbf{k})}$ . Then we find that  $\bar{f} \notin \bar{\mathfrak{M}}^2$  in  $\mathbf{k}[[X_0, \dots, X_d]]$ . The number of minimal generators of the maximal ideal of  $R/\pi_{W(\mathbf{k})}R$  is equal to

$$\dim_{\mathbf{k}} \mathfrak{m}/(\pi_{W(\mathbf{k})}R + \mathfrak{m}^2) = \dim_{\mathbf{k}} \mathfrak{m}/\mathfrak{m}^2,$$

due to  $\pi_{W(\mathbf{k})} \in \mathfrak{m}^2$ . So  $\bar{x}_0, \dots, \bar{x}_d$  are the minimal generators of the maximal ideal of  $R/\pi_{W(\mathbf{k})}R$ . Then there is a surjection

$$\mathbf{k}[[X_0, \dots, X_d]] \twoheadrightarrow R/\pi_{W(\mathbf{k})}R.$$

Now choose  $s$  such that

$$\bar{f} = \sum_{i=0}^{\infty} h_i X_s^i \in \mathbf{k}[[X_0, \dots, X_d]],$$

$h_i \in \mathbf{k}[[X_1, \dots, X_{s-1}, X_{s+1}, \dots, X_d]]$  for all  $i \geq 0$ , and  $h_1$  is a unit. Indeed, if  $h_1$  is not a unit, then since  $\bar{f} \notin \overline{\mathfrak{M}}^2$ ,  $h_0$  contains a nonzero linear term after presenting it as an (infinite) sum of homogeneous polynomials. Then replacing  $s$  by a suitable one, we can assume that  $h_1$  is a unit.

Taking reduction of  $\bar{f} = \sum_{i=0}^{\infty} h_i X_s^i$  by  $\bar{P}$ , we get  $\sum_{i=1}^{\infty} \bar{h}_i \bar{x}_s^i = -\bar{h}_0$  in  $R/\pi_{W(\mathbf{k})}R$ , and  $\bar{h}_1$  is a unit of  $R/\pi_{W(\mathbf{k})}R$ . Thus,

$$\bar{x}_s \cdot (\text{unit}) = -\bar{h}_0.$$

But this gives a contradiction to the fact that  $\bar{x}_0, \dots, \bar{x}_d$  are the minimal generators of the maximal ideal of  $R/\pi_{W(\mathbf{k})}R$  and that  $-\bar{h}_0 \in (\bar{x}_0, \dots, \bar{x}_{s-1}, \bar{x}_{s+1}, \dots, \bar{x}_d)$ . Hence,  $P \subseteq \pi_{W(\mathbf{k})}W(\mathbf{k})[[X_0, \dots, X_d]] + \mathfrak{M}^2$ .

In the next place, fix an arbitrary lifting  $\tilde{u} \in W(\mathbf{k})[[X_0, \dots, X_d]]$  of  $u$  under the map  $W(\mathbf{k})[[X_0, \dots, X_d]] \rightarrow R$ . We write:

$$\tilde{u} = \sum_{s_0, \dots, s_d} \left( \sum_{r=0}^{\infty} b_r^{(s_0, \dots, s_d)} \pi_{W(\mathbf{k})}^r \right) X_0^{s_0} \cdots X_d^{s_d},$$

where  $(s_0, \dots, s_d)$  signifies the multi-index, and all the elements  $b_r^{(s_0, \dots, s_d)}$  are the Teichmüller lifts. Since  $\tilde{u}$  is a unit,  $b_0^{(0, \dots, 0)} \neq 0$ . By lifting the relation  $\mathbf{x}_a = u\mathbf{x}_{a'}$  to  $W(\mathbf{k})[[X_0, \dots, X_d]]$ , we have

$$\sum_{i=0}^d a_i X_i \equiv \left( \sum_{s_0, \dots, s_d} \left( \sum_{r=0}^{\infty} b_r^{(s_0, \dots, s_d)} \pi_{W(\mathbf{k})}^r \right) X_0^{s_0} \cdots X_d^{s_d} \right) \left( \sum_{i=0}^d a'_i X_i \right) \pmod{P}.$$

Rewrite the above presentation as:

$$\begin{aligned} \sum_{i=0}^d (a_i - a'_i b_0^{(0, \dots, 0)}) X_i &\equiv \left( \sum_{s_0, \dots, s_d} \left( \sum_{r=1}^{\infty} b_r^{(s_0, \dots, s_d)} \pi_{W(\mathbf{k})}^r \right) X_0^{s_0} \cdots X_d^{s_d} \right) \left( \sum_{i=0}^d a'_i X_i \right) \\ &+ \left( \sum_{\substack{(s_0, \dots, s_d) \\ \neq (0, \dots, 0)}} b_0^{(s_0, \dots, s_d)} X_0^{s_0} \cdots X_d^{s_d} \right) \left( \sum_{i=0}^d a'_i X_i \right) \pmod{P}. \end{aligned}$$

Then by mapping the above formula to the quotient  $\mathbf{k}[[X_0, \dots, X_d]]$ , comparing the degrees on both sides, and then using the fact  $P \subseteq \pi_{W(\mathbf{k})}W(\mathbf{k})[[X_0, \dots, X_d]] + \mathfrak{M}^2$ , we find that

$$a_i = a'_i b_0^{(0, \dots, 0)} + \pi_{W(\mathbf{k})} \cdot v_i$$

for some  $v_i \in W(\mathbf{k})$ . However if  $v_i \neq 0$ , this implies that  $a_i$  is not a Teichmüller lift, which is false. So we have  $v_i = 0$  for all  $i$  and the following relation holds:

$$\begin{aligned} & \left( \sum_{s_0, \dots, s_d} \left( \sum_{r=1}^{\infty} b_r^{(s_0, \dots, s_d)} \pi_{W(\mathbf{k})}^r \right) X_0^{s_0} \cdots X_d^{s_d} \right) \left( \sum_{i=0}^d a_i' X_i \right) \\ & + \left( \sum_{\substack{(s_0, \dots, s_d) \\ \neq (0, \dots, 0)}} b_0^{(s_0, \dots, s_d)} X_0^{s_0} \cdots X_d^{s_d} \right) \left( \sum_{i=0}^d a_i' X_i \right) \in P. \end{aligned}$$

Since  $P$  is a prime ideal, we deduce that

$$\left( \sum_{s_0, \dots, s_d} \left( \sum_{r=1}^{\infty} b_r^{(s_0, \dots, s_d)} \pi_{W(\mathbf{k})}^r \right) X_0^{s_0} \cdots X_d^{s_d} + \sum_{\substack{(s_0, \dots, s_d) \\ \neq (0, \dots, 0)}} b_0^{(s_0, \dots, s_d)} X_0^{s_0} \cdots X_d^{s_d} \right) \in P$$

and thus  $\tilde{u} \equiv b_0^{(0, \dots, 0)} \pmod{P}$  and  $u \in W(\mathbf{k})$ . Hence  $u \in W(\mathbf{k})^\times$ , as desired.

**Case2:** Assume  $\pi_{W(\mathbf{k})} \notin \mathfrak{m}^2$ . Then taking  $x_0 = \pi_{W(\mathbf{k})}$ , we may consider the surjection  $W(\mathbf{k})[[X_1, \dots, X_d]] \twoheadrightarrow R$  ( $X_i \mapsto x_i$ ) and let  $P$  be its kernel. In this case, we claim that

$$P \subseteq \mathfrak{M}^2,$$

where  $\mathfrak{M} := (\pi_{W(\mathbf{k})}, X_1, \dots, X_d)$ . Indeed, any nonzero element  $f \in P$  can be presented in the form:

$$\sum_{s_1, \dots, s_d} \left( \sum_{r=0}^{\infty} b_r^{(s_1, \dots, s_d)} \pi_{W(\mathbf{k})}^r \right) X_1^{s_1} \cdots X_d^{s_d}.$$

Assume that  $f \notin \mathfrak{M}^2$ . Then we have  $\bar{f} \notin \overline{\mathfrak{M}}^2$  in  $R$  and the calculation of  $\bar{f} = 0$  in  $R$  as in **Case 1** shows that  $f$  contains a linear term with respect to either  $\pi_{W(\mathbf{k})}$ , or some  $X_i$ , which leads to a contradiction to the minimality of  $\pi_{W(\mathbf{k})}, x_1, \dots, x_d$ , as discussed in **Case 1**. Assume that  $\mathbf{x}_a = u \mathbf{x}_{a'}$  for some  $u \in R^\times$ . Then by applying the final step of **Case 1** together with the fact that  $P \subseteq \mathfrak{M}^2$ , we conclude that  $u \in W(\mathbf{k})^\times$ , as desired.  $\square$

Now Proposition 5.1 assures us that there are sufficiently many normal hyperplane sections for a normal local domain, which is as follows.

**Corollary 5.2.** *In addition to the hypothesis of Proposition 5.1, assume that  $\mathbf{k}$  is an infinite perfect field of characteristic  $p > 0$  and  $U \subseteq \mathbb{P}^d(\mathbf{k})$  is an infinite subset. Then  $\{\mathbf{x}_a R \mid a \in \langle \theta_{W(\mathbf{k})} \rangle(U)\}$  is an infinite set of mutually distinct height-one primes. In particular, we have*

$$\bigcap_{a \in \langle \theta_{W(\mathbf{k})} \rangle(U)} \mathbf{x}_a R = 0.$$

*Proof.* By the above proposition, the subset  $\langle \theta_{W(\mathbf{k})} \rangle(U) \subseteq \mathbb{P}^d(W(\mathbf{k}))$  parametrizes the set of height-one primes  $\{\mathbf{x}_a R\}$  and it is infinite, since  $U$  is so by assumption. Since  $R$  is a Noetherian domain, every nonzero nonunit element of  $R$  has only finitely many prime divisors (of height one). Therefore, we have the claim.  $\square$

The authors believe that Proposition 5.1 holds for any infinite residue field. Via the proof of the proposition, we obtain the following corollary, which is necessary in dealing with only the unramified case.

**Corollary 5.3.** *In addition to the notation of Theorem 4.4, assume that  $\pi_A$  is part of minimal generators of  $\mathfrak{m}$  and  $x_0 = \pi_A$ . Then  $\pi_A$  is part of minimal generators of the maximal ideal of  $R/\mathfrak{x}_a R$  for every  $a \in \mathbb{P}^d(A)$ .*

*Proof.* It is not necessary to assume that  $\mathbf{k}$  is a perfect field. Let  $A[[X_1, \dots, X_d]] \twoheadrightarrow R$  be a surjection with its kernel  $P$ , which is so constructed due to the assumption:  $\pi_A \notin \mathfrak{m}^2$ . Then  $P \subseteq (\pi_A, X_1, \dots, X_d)^2$ , as was seen in the proof of Proposition 5.1. Let  $\tilde{\mathfrak{x}}_a = a_0\pi_A + \sum_{i=1}^d a_i X_i$  be a lifting of  $\mathfrak{x}_a$ . Since every point of  $\mathbb{P}^d(A)$  is normalized,  $a_i \in A$  is a unit for some  $1 \leq i \leq d$ , we have an isomorphism:  $A[[X_1, \dots, X_d]]/(\tilde{\mathfrak{x}}_a) \simeq A[[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_d]]$  together with  $\pi_A \notin (\pi_A, X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_d)^2$ .

But since  $\overline{P} \subseteq (\pi_A, X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_d)^2$ , where  $\overline{P}$  is the image of  $P$  in the quotient  $A[[X_1, \dots, X_d]]/(\tilde{\mathfrak{x}}_a)$ , it follows that  $\pi_A$  is part of minimal generators of the maximal ideal of  $R/\mathfrak{x}_a R$ , as required.  $\square$

## 6. SERRE'S CONDITIONS $(R_n)$ AND $(S_n)$

Let us briefly state Bertini theorems for the punctured spectra of local rings in the case when  $R$  satisfies Serre's conditions. The essence for these cases already appears in the proof of the Bertini theorem for normal rings. In the following corollary, the quotient  $R/\mathfrak{x}_a R$  may fail to be reduced. As usual, we put  $X = \text{Spec } R - V(\mathfrak{m})$  and

$$\mathfrak{x}_a = \sum_{i=0}^d a_i x_i$$

for  $a = (a_0, \dots, a_d) \in \mathbb{A}^{d+1}(A)$ .

**Corollary 6.1.** *Suppose that  $(R, \mathfrak{m}, \mathbf{k})$  is a complete local reduced ring of mixed characteristic  $p > 0$ , that conditions (1) and (2) of Theorem 4.4 hold, and that the residue field  $\mathbf{k}$  is infinite. If  $R_{\mathfrak{p}}$  has Serre's  $(R_n)$  (resp.  $(S_n)$ ) for all  $\mathfrak{p} \in X$ , then there exists a Zariski dense open subset  $U \subseteq \mathbb{P}^d(\mathbf{k})$  such that for every*

$$a = (a_0 : \dots : a_d) \in \text{Sp}_A^{-1}(U),$$

*the quotient  $R_{\mathfrak{p}}/\mathfrak{x}_a R_{\mathfrak{p}}$  has  $(R_n)$  (resp.  $(S_n)$ ) for all  $\mathfrak{p} \in X \cap V(\mathfrak{x}_a)$ .*

*Proof.* We briefly sketch the proof of the corollary. Since  $R$  is complete local and reduced, the singular locus of  $R$  is non-empty. Thus, the set

$$Q_1 := \{\mathfrak{p} \in X \mid \mathfrak{p} \text{ is a minimal prime in } \text{Sing}(X)\}$$

is finite. Let

$$Q_2 := \{\mathfrak{p} \in X \mid \text{depth } R_{\mathfrak{p}} = n \text{ and } \dim R_{\mathfrak{p}} > n\},$$

which is also finite by ([1], lemma 3.2). The proof is similar to that of Theorem 4.4. So it suffices to avoid the union of finite set of prime ideals in  $Q_1 \cup Q_2$ . Namely, for any  $a = (a_0 : \cdots : a_d) \in \text{Sp}_A^{-1}(U)$ , the localization  $R_{\mathfrak{p}}/\mathfrak{x}_a R_{\mathfrak{p}}$  has  $(R_n)$  (resp.  $(S_n)$ ) for all  $\mathfrak{p} \in X \cap V(\mathfrak{x}_a)$  and  $\text{Reg}(R) \cap V(\mathfrak{x}_a) \subseteq \text{Reg}(R/\mathfrak{x}_a R)$ .  $\square$

The above proof also shows that the Bertini theorem holds for mixed Serre's conditions. That is, if  $R$  has  $(R_s) + (S_r)$ , then so does  $R_{\mathfrak{p}}/\mathfrak{x}_a R_{\mathfrak{p}}$  for all  $a = (a_0 : \cdots : a_d) \in \text{Sp}_A^{-1}(U)$  and all  $\mathfrak{p} \in X \cap V(\mathfrak{x}_a)$ . For instance, we obtain the Bertini theorem for reduced rings, since  $R$  is reduced  $\iff R$  has  $(R_0) + (S_1)$ . To be precise, we have the following version of Bertini theorem:

**Corollary 6.2.** *Suppose that  $(R, \mathfrak{m}, \mathbf{k})$  is a complete local normal domain of dimension 2 in mixed characteristic  $p > 0$ , that conditions (1) and (2) of Theorem 4.4 hold, and that the residue field  $\mathbf{k}$  is infinite. Then there exists a Zariski dense open subset  $U \subseteq \mathbb{P}^d(\mathbf{k})$  such that for every*

$$a = (a_0 : \cdots : a_d) \in \text{Sp}_A^{-1}(U),$$

*the quotient  $R/\mathfrak{x}_a R$  is a reduced ring of mixed characteristic  $p > 0$ .*

*Proof.* By the previous corollary,  $R_{\mathfrak{p}}/\mathfrak{x}_a R_{\mathfrak{p}}$  is reduced for all  $\mathfrak{p} \in X \cap V(\mathfrak{x}_a)$ . Since  $R$  is a local normal domain of dimension 2, we have  $\text{depth } R = 2$ . Hence  $R/\mathfrak{x}_a R$  is reduced. To make the quotient  $R/\mathfrak{x}_a R$  into a ring of mixed characteristic, it suffices to choose  $U$  such that  $\pi_A$  is not a zero divisor of  $R/\mathfrak{x}_a R$ . In fact, we may take  $\mathfrak{x}_a$  so that it avoids the union of finitely many minimal prime divisors of  $\pi_A R$ , since every system of parameters of  $R$  is a regular sequence. So the rest of the proof is similar to that of Theorem 4.4.  $\square$

*Remark 6.3.* In place of the hypothesis of Corollary 6.2, assume that  $R$  is only a domain. Then can one find  $\mathfrak{x}_a$  such that  $R/\mathfrak{x}_a R$  is a domain as well? In the mixed characteristic case, the answer to this question is not clear yet. But there is a two-dimensional complete normal local domain over  $\mathbb{C}$  without principal prime ideals at all (such an example is due to Laufer, as mentioned in [1]. However, an explicit example is not given there). In light of this, both Corollary 6.1 and Corollary 6.2 seem to be the best results.

## 7. CHARACTERISTIC IDEALS OF TORSION MODULES OVER NORMAL DOMAINS

Throughout this section, we assume that  $R$  is a Noetherian normal domain and  $M$  is a finitely generated torsion  $R$ -module. Then the localization of  $R$  at every height-one prime is a discrete valuation ring. We introduce an invariant of the module  $M$ . For an ideal  $I$  of  $R$ , let  $M[I]$  denote the maximal submodule of  $M$  which is annihilated by  $I$ . We follow the definition of characteristic ideals by Skinner-Urban as in [11]. For more results and

properties on characteristic ideals with its relation to the reflexive closure and the Fitting ideal, see § 9.

**Definition 7.1.** Let the notation be as above. Then the *characteristic ideal* is an ideal of  $R$  defined by

$$\text{char}_R(M) = \{x \in R \mid v_P(x) \geq \ell_{R_P}(M_P) \text{ for any height-one prime } P\},$$

where  $v_P(-)$  is the normalized valuation of  $R_P$ .

Since  $M$  is torsion,  $\ell_{R_P}(M_P) = 0$  for all but finitely many height-one primes  $\mathfrak{p}$  of  $R$  and it suffices to take only height-one primes in the support of  $M$  in the definition. When  $M$  is not a torsion  $R$ -module, we put  $\text{char}_R(M) = 0$ .

*Remark 7.2.* The formation of characteristic ideals does not commute with base change in general. For example, let  $R = \mathbb{Z}_p[[x, y]]$  and let  $M = R/xR$ . Then  $M$  is a torsion module and  $\text{char}_R(M) = xA$ . However, the  $R/xR$ -module  $M/xM$  is not torsion. Therefore,

$$0 = \text{char}_{R/xR}(M/xM) \subsetneq xR = \text{char}_R(M)R/xR$$

in this case. In general, even when  $M/xM$  is a torsion  $R/xR$ -module, it may happen that  $\text{char}_{R/xR}(M/xM) \neq \text{char}_R(M)R/xR$ , which is caused by the presence of pseudo-null submodules (see the definition below).

**Definition 7.3.** A finitely generated module  $M$  over a Noetherian normal domain  $R$  is *pseudo-null*, if  $M_{\mathfrak{p}} = 0$  for all height-one primes  $\mathfrak{p} \in \text{Spec } R$ . A homomorphism of  $R$ -modules  $f : M \rightarrow N$  is a *pseudo-isomorphism*, if both  $\ker(f)$  and  $\text{coker}(f)$  are pseudo-null modules.

The proof of the next lemma is found in ([6], Proposition 5.1.7).

**Lemma 7.4** (Structure Theorem). *Let  $M$  be a finitely generated torsion module over a Noetherian normal domain  $R$ . Then there exist a finite set of height-one primes  $\{P_i\}_{i \in I}$  (which is not necessarily a redundant set of height-one primes) and a set of natural numbers  $\{e_i\}_{i \in I}$  such that there is a homomorphism:*

$$f : M \rightarrow \bigoplus_{i \in I} R/P_i^{e_i}$$

*that is a pseudo-isomorphism. Moreover, both  $\{P_i\}_{i \in I}$  and  $\{e_i\}_{i \in I}$  are uniquely determined.*

Henceforth, we use the notation  $M \approx N$  to indicate that there is a pseudo-isomorphism between  $M$  and  $N$ . As mentioned before, the formation of characteristic ideals does not commute with base change in general, which can produce extra error terms.

**Proposition 7.5.** *Let  $M$  be a finitely generated torsion module over a Noetherian normal domain  $R$ . Let  $x \in R$  be an element which satisfies the following conditions:*

- (1)  $R/xR$  is a normal domain (which implies that  $xR$  is a prime ideal);
- (2)  $x$  is contained in no prime ideal of height one in the support of  $M$  (which implies that  $M/xM$  is a torsion  $R/xR$ -module).

Then we have:

$$\text{char}_{R/xR}(M/xM) = \left( \text{char}_{R/xR}(M[x]) \cdot \prod_{\substack{\text{ht } \mathfrak{p}=1, \\ \mathfrak{p} \in \text{Spec } R}} \mathfrak{p}(R/xR)^{\ell_{R\mathfrak{p}}(M_{\mathfrak{p}})} \right)^{**}.$$

*Proof.* Recall from Proposition 9.6 in Appendix, that if  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  is a short exact sequence of torsion  $R$ -modules, then  $\text{char}_R(M) = (\text{char}_R(L) \cdot \text{char}_R(N))^{**}$ , where  $(-)^{**}$  denotes the reflexive closure. The condition stated in the proposition implies that all relevant modules are torsion. First off, we claim that

$$\text{char}_{R/xR} \left( \bigoplus_{\text{ht } \mathfrak{p}=1} R/(\mathfrak{p} + xR)^{\ell_{R\mathfrak{p}}(M_{\mathfrak{p}})} \right) = \left( \prod_{\substack{\text{ht } \mathfrak{p}=1, \\ \mathfrak{p} \in \text{Spec } R}} \mathfrak{p}(R/xR)^{\ell_{R\mathfrak{p}}(M_{\mathfrak{p}})} \right)^{**}.$$

For the proof of this equality, let  $\{P_1, \dots, P_m\}$  be a set of all height-one primes of  $R/xR$  which contain  $\mathfrak{p}(R/xR)$ . Then the localization  $(R/xR)_{P_i}$  is a discrete valuation ring and

$$\mathfrak{p}(R/xR)_{P_i} = P_i^{\ell_{(R/xR)_{P_i}}((R/(\mathfrak{p}+xR))_{P_i})} (R/xR)_{P_i}$$

for all  $i$ . Then the above equality immediately follows from this. Thus, it suffices to prove the following equality:

$$\begin{aligned} & \left( \text{char}_{R/xR}(M/xM) \cdot \text{char}_{R/xR}(M[x])^{-1} \right)^{**} \\ &= \text{char}_{R/xR} \left( \bigoplus_{\substack{\text{ht } \mathfrak{p}=1, \\ \mathfrak{p} \in \text{Spec } R}} R/(\mathfrak{p} + xR)^{\ell_{R\mathfrak{p}}(M_{\mathfrak{p}})} \right), \end{aligned}$$

which is an element in the group of all reflexive fractional ideals of  $R$ . By considering all torsion  $R$ -modules such that  $x \in R$  is contained in no height-one prime ideal in their support, we show that both sides of the above formula is multiplicative on short exact sequences. Let

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

be a short exact sequence of such  $R$ -modules. Then since the function  $\ell(-)$  is additive, it follows that the right hand side of the above formula is multiplicative. On the other hand, there follows the exact sequence:

$$0 \rightarrow L[x] \rightarrow M[x] \rightarrow N[x] \rightarrow L/xL \rightarrow M/xM \rightarrow N/xN \rightarrow 0$$

by the snake lemma. If  $\mathfrak{p}$  is a height-two prime ideal of  $R$  containing  $x \in R$ , we may localize the above exact sequence at  $\mathfrak{p}$ , so it follows that the left hand side of the above formula is multiplicative as well.

Hence we are reduced to the case that  $M = R/\mathfrak{q}$  for a prime ideal  $\mathfrak{q}$  by the prime filtration argument. Assume first that  $\text{ht } \mathfrak{q} = 1$ . Then we clearly have  $\text{Supp}(M) = \{\mathfrak{q}\}$ ,  $M[x] = 0$ , and  $\ell_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) = 1$ , because  $M_{\mathfrak{q}}$  is a field. Now the formula is obviously true. Next assume that  $\text{ht } \mathfrak{q} > 1$ . Then it is easy to see that both sides of the formula are equal to a unit ideal, which completes the proof.  $\square$

## 8. APPLICATIONS TO CHARACTERISTIC IDEALS

Our final goal is to prove Theorem 8.7. The aim of the main theorem is to establish some techniques, which enables us to study Iwasawa Main Conjecture (see [9]), so we need to deal with local rings with finite residue field. Let  $(R, \mathfrak{m}, \mathbb{F})$  be a complete normal local domain of mixed characteristic with finite residue field. In other words,  $R$  is the integral closure of  $\mathbb{Z}_p[[z_1, \dots, z_n]]$  in a finite field extension of the field of fractions of  $\mathbb{Z}_p[[z_1, \dots, z_n]]$ . Let us recall the set-up of Corollary 4.6 and prove some preliminary results.

Recall that  $R_{\widehat{\mathcal{O}}_{\text{ur}}} := R \widehat{\otimes}_{\mathcal{O}} \widehat{\mathcal{O}}_{\text{ur}}$  with its coefficient ring  $\widehat{\mathcal{O}}_{\text{ur}}$ . Then if  $\text{depth } R \geq 3$ , the complete local ring  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$  fits into the hypothesis of Theorem 4.4. The Teichmüller mapping  $\overline{\mathbb{F}} \rightarrow \widehat{\mathcal{O}}_{\text{ur}}$  induces  $\langle \theta_{\widehat{\mathcal{O}}_{\text{ur}}} \rangle : \mathbb{P}^d(\overline{\mathbb{F}}) \rightarrow \mathbb{P}^d(\widehat{\mathcal{O}}_{\text{ur}})$  (there is an inclusion  $\mathbb{P}^d(\mathcal{O}_{\text{ur}}) \subseteq \mathbb{P}^d(\widehat{\mathcal{O}}_{\text{ur}})$ ).

To establish the fundamental theorem for characteristic ideals, we need to relate the torsion  $R$ -modules to the  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -modules and then descend to  $R$  by faithful flatness. The advantage of working with  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$  is that the residue field is infinite and perfect. We introduce some notation. Denote by  $\text{Fitt}_A(M)$  the Fitting ideal of the  $A$ -module  $M$ . We make free use of results and notation from Appendix.

**Definition 8.1.** Under the notation as above, fix a set of minimal generators  $x_1, \dots, x_d$  of the unique maximal ideal of  $R$ . Then we set

$$\mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}} := \{\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \mid a = (a_0 : \dots : a_d) \in \langle \theta_{\widehat{\mathcal{O}}_{\text{ur}}} \rangle(U)\}$$

for the mapping  $\langle \theta_{\widehat{\mathcal{O}}_{\text{ur}}} \rangle : \mathbb{P}^d(\overline{\mathbb{F}}) \rightarrow \mathbb{P}^d(\widehat{\mathcal{O}}_{\text{ur}})$ . For a finitely generated torsion  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -module  $M$ , we define a subset  $\mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M) \subseteq \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}$  which consists of all height-one primes  $\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}$  such that the following conditions are satisfied:

- (A)  $M/\mathbf{x}_a M$  is a torsion  $R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -module.
- (B) The following equalities of ideals hold in  $R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}$ :

$$\begin{aligned} \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M/\mathbf{x}_a M) &= (\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M) R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}})^{**} \\ &= (\text{Fitt}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(\bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i}) R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}})^{**}, \end{aligned}$$

where  $(-)^* = \text{Hom}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(-, R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}})$  and  $M \approx \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i}$  is a fundamental pseudo-isomorphism.



When the base ring  $R$  is complete regular, this definition actually coincides with the format as given in ([8], Definition 3.2.). The way of interpreting the condition **(B)** is that one wants to see the characteristic ideals through Fitting ideals as an intermediate invariant. Note that all of three ideals appearing in **(B)** may differ in general.

**Lemma 8.2.** *Let the notation and the hypothesis be as in Corollary 4.6 and let  $I \subseteq R_{\widehat{\mathcal{O}_{\text{ur}}}}$  be an ideal and consider the natural injection  $I \rightarrow I^{**}$  with its cokernel  $N$ . Then there exists a finite set  $\{Q_i\}_{1 \leq i \leq \ell}$  consisting of height-two primes of  $R_{\widehat{\mathcal{O}_{\text{ur}}}}$  such that for*

$$\mathbf{x}_a R_{\widehat{\mathcal{O}_{\text{ur}}}} \in \bigcap_{1 \leq i \leq \ell} \mathcal{L}_{\widehat{\mathcal{O}_{\text{ur}}}}(R_{\widehat{\mathcal{O}_{\text{ur}}}}/Q_i)$$

and for all  $P \in \text{Spec } R_{\widehat{\mathcal{O}_{\text{ur}}}}$  with the property that  $\mathbf{x}_a \in P$  and  $\text{ht } P \leq 2$ , we have  $N_P = 0$ .

*Proof.* By definition, the module  $N$  is supported on a closed subset of codimension two in  $\text{Spec } \widehat{R^{\text{sh}}}$ . So there are only finitely many height-two prime ideals contained in  $\text{Supp } N$ . Hence it is sufficient to choose  $Q_1, \dots, Q_\ell$  as height-two prime ideals in  $\text{Supp } N$ .  $\square$

In the following remark, let us examine when equalities occur between various ideals in the condition **(B)**.

*Remark 8.3.* Suppose  $M$  is a finitely generated torsion  $A$ -module. Then we have

$$\text{Fitt}_A(M)^{**} = \text{char}_A(M)$$

(see Proposition 9.6 in Appendix). Take a fundamental pseudo-isomorphism:

$$M \approx \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}_{\text{ur}}}}/P_i^{e_i}$$

for a (not necessarily redundant) finite set of height-one primes  $\{P_i\}$  of  $R_{\widehat{\mathcal{O}_{\text{ur}}}}$ . Choose  $\mathbf{x}_a \in R_{\widehat{\mathcal{O}_{\text{ur}}}}$  such that  $R_{\widehat{\mathcal{O}_{\text{ur}}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}_{\text{ur}}}}$  is normal and  $M/\mathbf{x}_a M$  is a torsion  $R_{\widehat{\mathcal{O}_{\text{ur}}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}_{\text{ur}}}}$ -module. In particular, the multiplication map:

$$\bigoplus_{i=1}^m R_{\widehat{\mathcal{O}_{\text{ur}}}}/P_i^{e_i} \xrightarrow{\mathbf{x}_a} \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}_{\text{ur}}}}/P_i^{e_i}$$

is injective. Then since  $\text{Fitt}_B(M \otimes_A B) = \text{Fitt}_A(M)B$  for any Noetherian  $A$ -algebra  $B$ , letting  $B = R_{\widehat{\mathcal{O}_{\text{ur}}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}_{\text{ur}}}}$ , it follows that

$$\begin{aligned} (\text{char}_{R_{\widehat{\mathcal{O}_{\text{ur}}}}}(M) R_{\widehat{\mathcal{O}_{\text{ur}}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}_{\text{ur}}}})^{**} &= (\text{Fitt}_{R_{\widehat{\mathcal{O}_{\text{ur}}}}}(\bigoplus_{i=1}^m R_{\widehat{\mathcal{O}_{\text{ur}}}}/P_i^{e_i}) R_{\widehat{\mathcal{O}_{\text{ur}}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}_{\text{ur}}}})^{**} \\ &= \text{Fitt}_{R_{\widehat{\mathcal{O}_{\text{ur}}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}_{\text{ur}}}}}(\bigoplus_{i=1}^m R_{\widehat{\mathcal{O}_{\text{ur}}}}/(\mathbf{x}_a R_{\widehat{\mathcal{O}_{\text{ur}}}} + P_i^{e_i}))^{**} \end{aligned}$$

as long as

$$\mathbf{x}_a R_{\widehat{\mathcal{O}_{\text{ur}}}} \in \bigcap_{1 \leq i \leq \ell} \mathcal{L}_{\widehat{\mathcal{O}_{\text{ur}}}}(R_{\widehat{\mathcal{O}_{\text{ur}}}}/Q_i),$$

where  $\{Q_i\}_{1 \leq i \leq \ell}$  is specified as in Lemma 8.2 (of course, we have assumed the normality of  $R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}$  as well as the torsion property of  $M/\mathbf{x}_a M$ ). It also yields that

$$(\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M)R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}})^{**} = \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(\bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/(\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} + P_i^{e_i})),$$

which we will use below.

Some preliminary results from [8] may be proved in a similar way over general normal domains with necessary modifications.

**Lemma 8.4.** *Under the notation and the hypothesis as in Corollary 4.6, assume that  $M$  is a finitely generated torsion  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -module. Then the following assertions hold:*

(1) *The set  $\mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M)$  is identified with the intersection:*

$$\{\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \mid M/\mathbf{x}_a M \text{ is a torsion } R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}\text{-module}\} \cap \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\text{null}}) \cap \bigcap_{1 \leq i \leq \ell} \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/Q_i),$$

where  $M_{\text{null}}$  is the maximal pseudo-null submodule of  $M$  and  $\{Q_i\}_{1 \leq i \leq \ell}$  is a set of height-two primes as stated in Lemma 8.2.

(2) *Assume that  $N$  is a finitely generated pseudo-null  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -module and  $\{Q'_i\}_{1 \leq i \leq k}$  is a set of all associated prime ideals of height two for the module  $N$ . Then we have:*

$$\mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(N) = \bigcap_{1 \leq i \leq k} \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/Q'_i).$$

*Proof.* (1): This is taken from ([8], Lemma 3.4.), but we give its proof, as it requires some modifications. Let  $M_{\text{null}}$  be the maximal pseudo-null submodule of  $M$ . Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M/M_{\text{null}} & \longrightarrow & \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i} & \longrightarrow & N \longrightarrow 0 \\ & & \mathbf{x}_a \downarrow & & \mathbf{x}_a \downarrow & & \mathbf{x}_a \downarrow \\ 0 & \longrightarrow & M/M_{\text{null}} & \longrightarrow & \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i} & \longrightarrow & N \longrightarrow 0 \end{array}$$

for a (not necessarily redundant) set of height-one primes  $\{P_i\}_{1 \leq i \leq m}$  of  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$  and  $N$  is a pseudo-null module.

Assume that  $M/\mathbf{x}_a M$  is a torsion  $R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -module for  $\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \bigcap_{1 \leq i \leq \ell} \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/Q_i)$  with the notation as in Lemma 8.2. Then the map

$$\bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i} \xrightarrow{\mathbf{x}_a} \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i}$$

is injective. So the snake lemma yields the following exact sequence:

$$0 \rightarrow N[\mathbf{x}_a] \rightarrow M/(\mathbf{x}_a M + M_{\text{null}}) \rightarrow \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/(\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} + P_i^{e_i}) \rightarrow N/\mathbf{x}_a N \rightarrow 0.$$

There is a short exact sequence:

$$0 \longrightarrow N[\mathbf{x}_a] \longrightarrow N \xrightarrow{\mathbf{x}_a} N \longrightarrow N/\mathbf{x}_a N \longrightarrow 0$$

of  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -modules, whereas both  $N[\mathbf{x}_a]$  and  $N/\mathbf{x}_a N$  are naturally  $R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -modules. Localizing this sequence at all height-two primes  $P$  of  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$  containing  $\mathbf{x}_a$ , a length computation for the sequence localized at  $P$  reveals that

$$\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M/(\mathbf{x}_a M + M_{\text{null}})) = \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}\left(\bigoplus_{1 \leq i \leq m} R_{\widehat{\mathcal{O}}_{\text{ur}}}/(\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} + P_i^{e_i})\right).$$

On the other hand, Remark 8.3 shows that

$$(\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M)R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}})^{**} = \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}\left(\bigoplus_{1 \leq i \leq m} R_{\widehat{\mathcal{O}}_{\text{ur}}}/(\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} + P_i^{e_i})\right).$$

Finally, since the multiplication on  $M/M_{\text{null}}$  by  $\mathbf{x}_a$  is injective, we have an exact sequence:

$$0 \rightarrow M_{\text{null}}/\mathbf{x}_a M_{\text{null}} \rightarrow M/\mathbf{x}_a M \rightarrow M/(\mathbf{x}_a M + M_{\text{null}}) \rightarrow 0.$$

Taking characteristic ideals to this exact sequence, we find that

$$\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M) \iff \mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\text{null}}),$$

because of the condition **(B)**. This completes the proof of (1).

(2): This part is done in ([8], Lemma 3.5 together with Lemma 3.1) in the case  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$  is regular, so we leave the proof with necessary modifications to the reader.  $\square$

The next objective is to identify the set  $\mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M)$  for a finitely generated torsion  $R$ -module  $M$ . For the proof of the main theorem, we need to make sure that there are sufficiently many specializations that are normal, and then study characteristic ideals by combining Proposition 5.1, Lemma 8.2, and Lemma 8.4.

**Lemma 8.5.** *Under the notation and the hypothesis as in Corollary 4.6, assume that  $M$  is a finitely generated torsion  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -module. Then we have the following assertions:*

- (1) *The subset  $\mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M) \subseteq \langle \theta_{\widehat{\mathcal{O}}_{\text{ur}}} \rangle(\mathcal{U})$  may be identified with a non-empty open subset of  $\mathbb{P}^d(\overline{\mathbb{F}})$  under the mapping  $\langle \theta_{\widehat{\mathcal{O}}_{\text{ur}}} \rangle : \mathbb{P}^d(\overline{\mathbb{F}}) \rightarrow \mathbb{P}^d(\widehat{\mathcal{O}}_{\text{ur}})$ . In particular, it is infinite.*
- (2) *There exists an infinite sequence  $\{\mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}\}_{i \in \mathbb{N}} \subseteq \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M)$  such that the union of the set of all minimal prime divisors:*

$$\bigcup_{i \in \mathbb{N}} \text{Min}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(P + \mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}})$$

*is infinite, where  $P$  appears as one of the components in  $\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M)$ .*

*Proof.* (1): By Lemma 8.4, it follows that  $\mathbf{x}_a \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\text{null}})$  if and only if  $\mathbf{x}_a \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}$  avoids the union of all height-two primes  $Q'_1, \dots, Q'_k$  contained in  $\text{Ass}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M_{\text{null}})$ . Let  $Q_1, \dots, Q_\ell$  be as given in Lemma 8.2. By the assumption that  $\dim R \geq 3$ , all those primes together with the set of height-one primes  $P_1, \dots, P_m$  in  $\text{Supp } M$  are strictly contained in the maximal ideal of  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ . So it remains to determine the non-empty Zariski open subset of  $\mathbb{P}^d(\overline{\mathbb{F}})$  with the required properties. In other words, it suffices to choose  $\mathbf{x}_a$  such that  $\mathbf{x}_a \notin (\bigcup_{1 \leq i \leq m} P_i) \cup (\bigcup_{1 \leq i \leq k} Q'_i) \cup (\bigcup_{1 \leq i \leq \ell} Q_i)$ . However, this part may be carried out in the same way as the final step of Theorem 4.3, so we leave the detail to the reader.

(2): We shall be done by completing inductive steps using (1) as follows. Let  $P$  be a fixed divisor of  $\text{char}_{\widehat{R}^{\text{sh}}}(M)$ . Since  $P$  is a non-maximal prime ideal of  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ , it corresponds to a non-empty Zariski open subset of  $\mathbb{P}(\overline{\mathbb{F}})$  and we may find  $\mathbf{x}_{a_0} \in \mathcal{L}_{\widehat{\mathcal{O}}^{\text{sh}}}(M)$ , which is not contained in  $P$ . Then this initial choice satisfies our requirement.

Choose  $\mathbf{x}_{a_1}$  so that it is not contained in a finite set of prime divisors  $\text{Min}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(P + \mathbf{x}_{a_0} R_{\widehat{\mathcal{O}}_{\text{ur}}})$  (which is possible due to  $\dim R \geq 3$ ). Next, choose  $\mathbf{x}_{a_2}$  so that it is not contained in the union of the finite set of prime divisors:

$$\text{Min}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(P + \mathbf{x}_{a_0} R_{\widehat{\mathcal{O}}_{\text{ur}}}) \cup \text{Min}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(P + \mathbf{x}_{a_1} R_{\widehat{\mathcal{O}}_{\text{ur}}})$$

Hence we may keep this process and get a sequence  $\mathbf{x}_{a_0}, \mathbf{x}_{a_1}, \mathbf{x}_{a_2}, \dots$  satisfying the required properties.  $\square$

Let  $M$  be a module over a complete local domain  $R$  with  $\mathcal{O}$  its coefficient ring and let  $\mathcal{O} \rightarrow \mathcal{O}'$  be a torsion-free extension of complete discrete valuation rings. Then we set  $M_{\mathcal{O}'} := M \widehat{\otimes}_{\mathcal{O}} \mathcal{O}'$ , which coincides with our previous notation. Let  $M$  be a finitely generated torsion  $R$ -module. Then

$$\text{char}_{R_{\mathcal{O}'}}(M_{\mathcal{O}'}) = \text{char}_R(M) R_{\mathcal{O}'},$$

which may be verified directly from the definition.

**Lemma 8.6.** *Let  $M, N$  be finitely generated torsion  $R$ -modules. Then we have*

$$\text{char}_R(M) \subseteq \text{char}_R(N) \iff \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}) \subseteq \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}).$$

*Proof.* The implication  $\Rightarrow$  is obvious. So let us prove the other implication. Since  $R \rightarrow R_{\widehat{\mathcal{O}}_{\text{ur}}}$  is ind-étale, it suffices to show that

$$\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M) \subseteq \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(N) \iff \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}) \subseteq \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}).$$

Note that  $R_{\widehat{\mathcal{O}}_{\text{ur}}} \rightarrow R_{\widehat{\mathcal{O}}_{\text{ur}}}$  is faithfully flat, with trivial residue field extension. Then the claim follows from this.  $\square$

Let us prove the following main theorem on the inclusion of characteristic ideals, which is fundamental for the study of Iwasawa theory and Euler system theory over general normal domains.

**Theorem 8.7** (Control Theorem for Characteristic Ideals). *With the notation and the hypothesis as in Corollary 4.6, assume that  $M$  and  $N$  are finitely generated torsion  $R$ -modules. Then the following statements are equivalent:*

- (1)  $\text{char}_R(M) \subseteq \text{char}_R(N)$ .
- (2) *For all but finitely many height-one primes:*

$$\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}) \cap \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}),$$

*there exists a finite étale extension of discrete valuation rings  $\mathcal{O} \rightarrow \mathcal{O}'$  such that we have  $\mathbf{x}_a \in R_{\mathcal{O}'}$  and*

$$\text{char}_{R_{\mathcal{O}'}/\mathbf{x}_a R_{\mathcal{O}'}}(M_{\mathcal{O}'}/\mathbf{x}_a M_{\mathcal{O}'}) \subseteq \text{char}_{R_{\mathcal{O}'}/\mathbf{x}_a R_{\mathcal{O}'}}(N_{\mathcal{O}'}/\mathbf{x}_a N_{\mathcal{O}'}).$$

- (3) *For all but finitely many height-one primes:*

$$\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}) \cap \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}),$$

*we have*

$$\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a M_{\widehat{\mathcal{O}}_{\text{ur}}}) \subseteq \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a N_{\widehat{\mathcal{O}}_{\text{ur}}}).$$

Note that  $\dim R \geq 3$  holds automatically, due to the hypothesis  $\text{depth } R \geq 3$ .

*Proof.* The implications (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) are obvious. So it remains to prove (3)  $\Rightarrow$  (1). By Lemma 8.6, it suffices to show that

$$\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}) \subseteq \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}).$$

Take fundamental pseudo-isomorphisms for  $M$  and  $N$ :

$$M \rightarrow \bigoplus_i R_{\widehat{\mathcal{O}}_{\text{ur}}} / P_i^{e_i} \text{ (resp. } N \rightarrow \bigoplus_j R_{\widehat{\mathcal{O}}_{\text{ur}}} / Q_j^{f_j})$$

for a (not necessarily redundant) finite set of height-one primes  $\{P_i\}$  (resp.  $\{Q_j\}$ ) of  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ . Put  $I_M := (\prod_i P_i^{e_i})^{**}$  and  $I_N := (\prod_j Q_j^{f_j})^{**}$  and the condition **(B)** allows one to assume that

$$M = \bigoplus_i R_{\widehat{\mathcal{O}}_{\text{ur}}} / P_i^{e_i} \text{ (resp. } N = \bigoplus_j R_{\widehat{\mathcal{O}}_{\text{ur}}} / Q_j^{f_j}).$$

To simplify the notation, assume that  $\{P_i\}$  (resp.  $\{Q_j\}$ ) is a redundant set of prime ideals and all relevant modules are defined over  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ . We make a step-by-step study to complete the proof. Let

$$\{\mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}\}_{i \in \mathbb{N}} \subseteq \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}) \cap \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}})$$

be any infinite sequence of mutually distinct primes of  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$  satisfying the condition (3). In particular, we have  $\bigcap_{i \in \mathbb{N}} \mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}} = 0$ .

**Step1:** Let us establish  $\text{Supp}_{\text{ht}=1} N \subseteq \text{Supp}_{\text{ht}=1} M$ , where  $\text{Supp}_{\text{ht}=1}(-)$  is the set of height-one primes contained in the support of a module. By assumption, we find that

$$(I_M(R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}))^{**} \subseteq (I_N(R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}))^{**}$$

for all  $i \in \mathbb{N}$ . Fix a divisor  $Q_k$ . Then for every fixed  $i \in \mathbb{N}$ , in particular, we have

$$I_M \subseteq \bigcap_{\mathfrak{a} \in \Lambda_i} \mathfrak{a},$$

where  $\Lambda_i$  is the set of symbolic powers associated to all minimal prime ideals of

$$\text{Ass}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/(Q_k + \mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}).$$

The image of the natural map

$$\bigcap_{\mathfrak{a} \in \Lambda_i} \mathfrak{a} \rightarrow R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}$$

is the reflexive closure of  $Q_k(R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}})$ . Moreover, we have an injection:

$$R_{\widehat{\mathcal{O}}_{\text{ur}}}/\bigcap_{i \in \mathbb{N}} \bigcap_{\mathfrak{a} \in \Lambda_i} \mathfrak{a} \hookrightarrow \prod_{i \in \mathbb{N}} (R_{\widehat{\mathcal{O}}_{\text{ur}}}/\bigcap_{\mathfrak{a} \in \Lambda_i} \mathfrak{a}).$$

Since  $Q_k$  is a prime ideal and since we could have chosen the set  $\{x_i\}_{i \in \mathbb{N}}$  such that the set of minimal primes divisors

$$\bigcup_{i \in \mathbb{N}} \text{Min}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(Q_k + \mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}) = \bigcup_{i \in \mathbb{N}} \bigcup_{\mathfrak{a} \in \Lambda_i} \text{Min}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(\mathfrak{a})$$

is infinite in view of Lemma 8.5, it follows that

$$Q_k = \bigcap_{i \in \mathbb{N}} \bigcap_{\mathfrak{a} \in \Lambda_i} \mathfrak{a},$$

and thus we have  $I_M \subseteq Q_k$ . Since  $Q_k$  is arbitrary,  $I_M \subseteq (\prod_j Q_j)^{**}$ , or equivalently,  $\text{Supp}_{\text{ht}=1} N \subseteq \text{Supp}_{\text{ht}=1} M$ .

**Step2:** In this step, we deal with multiplicities of divisors in the characteristic ideal and we complete this step by induction on the number of divisors appearing in  $I_M$ .

First, assume  $I_M = (P^e)^{**}$  for  $e \geq 1$ . Then we have  $\text{Supp}_{\text{ht}=1} N = \emptyset$  or  $\{P\}$  because of **Step1**. If  $\text{Supp}_{\text{ht}=1} N = \emptyset$ , there is nothing to prove. So assume  $\text{Supp}_{\text{ht}=1} N = \{P\}$ . Both  $M$  and  $N$  are assumed to be fundamental torsion  $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -modules, thus  $M[\mathbf{x}_{a_i}]$  and  $N[\mathbf{x}_{a_i}]$  are trivial modules and Proposition 7.5 yields that  $\ell_{(R_{\widehat{\mathcal{O}}_{\text{ur}}})_P}(M_P) \geq \ell_{(R_{\widehat{\mathcal{O}}_{\text{ur}}})_P}(N_P)$ .

In the general case, we prove by contradiction and thus, assume that  $I_M \not\subseteq I_N$ . Then this implies that we have  $e_k < f_k$  for some  $k$ . Put

$$\widetilde{I}_M := (P_k^{-e_k} \cdot I_M)^{**} \text{ (resp. } \widetilde{I}_N := (P_k^{-e_k} \cdot I_N)^{**}),$$

which are both integral reflexive ideals. There is a short exact sequence:

$$0 \rightarrow \tilde{I}_M/I_M \rightarrow R_{\widehat{\mathcal{O}_{\text{ur}}}}/I_M \rightarrow R_{\widehat{\mathcal{O}_{\text{ur}}}}/\tilde{I}_M \rightarrow 0 \text{ (resp. } 0 \rightarrow \tilde{I}_N/I_N \rightarrow R_{\widehat{\mathcal{O}_{\text{ur}}}}/I_N \rightarrow R_{\widehat{\mathcal{O}_{\text{ur}}}}/\tilde{I}_N \rightarrow 0)$$

and it is clear that

$$\text{char}_{R_{\widehat{\mathcal{O}_{\text{ur}}}}}(\tilde{I}_M/I_M) = \text{char}_{R_{\widehat{\mathcal{O}_{\text{ur}}}}}(\tilde{I}_N/I_N) = (P_k^{e_k})^{**},$$

which induces the following short exact sequences by the snake lemma:

$$0 \rightarrow \tilde{I}_M/(I_M, \mathbf{x}_{a_i}\tilde{I}_M) \rightarrow R_{\widehat{\mathcal{O}_{\text{ur}}}}/(I_M, \mathbf{x}_{a_i}) \rightarrow R_{\widehat{\mathcal{O}_{\text{ur}}}}/(\tilde{I}_M, \mathbf{x}_{a_i}) \rightarrow 0$$

and

$$0 \rightarrow \tilde{I}_N/(I_N, \mathbf{x}_{a_i}\tilde{I}_N) \rightarrow R_{\widehat{\mathcal{O}_{\text{ur}}}}/(I_N, \mathbf{x}_{a_i}) \rightarrow R_{\widehat{\mathcal{O}_{\text{ur}}}}/(\tilde{I}_N, \mathbf{x}_{a_i}) \rightarrow 0.$$

Taking characteristic ideals, we get by the assumption (3)

$$\text{char}_{R_{\widehat{\mathcal{O}_{\text{ur}}}/\mathbf{x}_{a_i}R_{\widehat{\mathcal{O}_{\text{ur}}}}}(\tilde{I}_M, \mathbf{x}_{a_i})} (R_{\widehat{\mathcal{O}_{\text{ur}}}}/(\tilde{I}_M, \mathbf{x}_{a_i})) \subseteq \text{char}_{R_{\widehat{\mathcal{O}_{\text{ur}}}/\mathbf{x}_{a_i}R_{\widehat{\mathcal{O}_{\text{ur}}}}}(\tilde{I}_N, \mathbf{x}_{a_i})} (R_{\widehat{\mathcal{O}_{\text{ur}}}}/(\tilde{I}_N, \mathbf{x}_{a_i})).$$

Since the number of primes divisors in  $\text{Ass}_{R_{\widehat{\mathcal{O}_{\text{ur}}}}}(R_{\widehat{\mathcal{O}_{\text{ur}}}}/\tilde{I}_M)$  is just one less than that of components of prime divisors in  $I_M$ , the induction hypothesis on  $\tilde{I}_M$  yields

$$\text{char}_{R_{\widehat{\mathcal{O}_{\text{ur}}}}}(R_{\widehat{\mathcal{O}_{\text{ur}}}}/\tilde{I}_M) \subseteq \text{char}_{R_{\widehat{\mathcal{O}_{\text{ur}}}}}(R_{\widehat{\mathcal{O}_{\text{ur}}}}/\tilde{I}_N).$$

However, we deduce from these observations that  $I_M \subseteq I_N$ , which is a contradiction to our assumption  $I_M \not\subseteq I_N$ . Hence, we obtain  $I_M \subseteq I_N$ , as desired.  $\square$

*Remark 8.8.* It is worth pointing out that Theorem 8.7 holds for complete normal local rings of mixed characteristic with arbitrary infinite perfect residue field as well. More precisely, it can be proven that  $\text{char}_R(M) \subseteq \text{char}_R(N) \iff \text{char}_{R/xR}(M/xM) \subseteq \text{char}_{R/xR}(N/xN)$  for sufficiently many  $x \in R$ .

In this article, we presented an application of Bertini theorem to characteristic ideals. However, we believe that the main theorem has more interesting applications such as the study of the restriction map on divisor class (Chow) groups.

## 9. APPENDIX

In this appendix, we study the relationship between Fitting ideals and characteristic ideals. For Fitting ideals, we refer the reader to Northcott's book [7], but we review the basic part of the theory. Throughout, we assume that  $R$  is a Noetherian ring and  $M$  is a finitely generated  $R$ -module.

**Definition 9.1** (Fitting ideal). Let the notation be as above and assume that

$$F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

is a finite free resolution of the  $R$ -module  $M$ , where the mapping  $F_1 \rightarrow F_0$  is defined via a  $m \times n$ -matrix  $X$  with  $\text{rank}(F_1) = n$  and  $\text{rank}(F_0) = m$ . Then  $\text{Fitt}_R(M)$  is defined as an ideal of  $R$  generated by all  $m$ -minors of  $X$ .

The Fitting ideal does not depend on the choice of a free resolution and it enjoys the following properties.

**Proposition 9.2.** *Let  $M$  be a finitely generated module over a Noetherian ring  $R$ . Then we have the following properties.*

- (1) *Let  $I \subseteq R$  be an ideal. Then  $\text{Fitt}_R(R/I) = I$ .*
- (2) *Let  $S$  be any Noetherian  $R$ -algebra. Then  $\text{Fitt}_S(M \otimes_R S) = \text{Fitt}_R(M)S$ .*
- (3) *Let  $\text{Ann}_R(M)$  be the annihilator of the  $R$ -module  $M$ . Then*

$$\text{Fitt}_R(M) \subseteq \text{Ann}_R(M).$$

- (4) *If  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  is a short exact sequence of  $R$ -modules, then*

$$\text{Fitt}_R(L) \cdot \text{Fitt}_R(N) \subseteq \text{Fitt}_R(M).$$

- (5) *Assume that  $R$  is a discrete valuation ring with its uniformizing parameter  $b$  and  $M$  is a torsion  $R$ -module. Then  $\text{Fitt}_R(M) = (b)^{\ell_R(M)}$ .*

*Proof.* These facts are all well known. For (5), it simply follows from the elementary divisors of modules over a principal ideal domain.  $\square$

For a Noetherian domain  $R$  and an  $R$ -module  $M$ , let  $M^* := \text{Hom}_R(M, R)$ , the dual of  $M$ . We say that  $M^{**}$  is the *reflexive closure* of  $M$ . Then we have the following lemma.

**Lemma 9.3.** *Let  $R$  be a Noetherian domain and let  $I$  be a fractional ideal of  $R$ . Then the reflexive closure  $I^{**}$  is naturally regarded as a fractional ideal of  $R$ .*

*Proof.* By assumption, there exists  $\alpha \in R$  such that  $I \simeq \alpha \cdot I \subseteq R$ . Let  $J := \alpha \cdot I$ , an ideal of  $R$ . The short exact sequence  $0 \rightarrow J \rightarrow R \rightarrow R/J \rightarrow 0$  induces a short exact sequence

$$0 = \text{Hom}_R(R/J, R) \rightarrow R \rightarrow \text{Hom}_R(J, R) \rightarrow N \rightarrow 0,$$

with  $N \subseteq \text{Ext}_R^1(R/J, R)$  cokernel of  $R \rightarrow \text{Hom}_R(J, R)$ . Then applying  $\text{Hom}_R(-, R)$ , we get an exact sequence

$$0 = \text{Hom}_R(N, R) \rightarrow J^{**} \rightarrow R,$$

because  $J \cdot N = 0$ . This implies that  $J^{**}$  is an ideal of  $R$ . Then  $I^{**} = \alpha^{-1} \cdot J^{**}$  is a fractional ideal.  $\square$



Note that even when  $I$  and  $J$  are reflexive, the product  $I \cdot J$  need not be reflexive. Any principal ideal is reflexive. Let  $I$  be an ideal of a normal domain  $R$ . Then we recall the following fact:

$$I^{**} = \bigcap_P I_P,$$

where  $P$  ranges over all height-one primes of  $R$ . The natural inclusion  $I \rightarrow I^{**}$  is a pseudo-isomorphism, since  $I_P \rightarrow (I^{**})_P = (I_P)^{**}$  for every height-one prime  $P \subseteq R$ . The following lemma explains the naturality of reflexive ideals and gives a way to investigate the inclusion relation between characteristic ideals.

**Lemma 9.4.** *Let  $R$  be a Noetherian normal domain and let  $I$  and  $J$  be reflexive ideals. Then  $I \subseteq J$  if and only if  $v_P(I) \geq v_P(J)$  for a valuation  $v$  attached to every height-one prime  $P$  of  $R$ . In particular, the only reflexive integral ideal containing a prime ideal of  $R$  properly is  $R$  itself.*

We defined characteristic ideals as reflexive ideals and this is natural from the viewpoint of Iwasawa theory, because the most interesting arithmetic information may be captured at height-one primes. For finitely generated torsion  $R$ -modules  $M, N$ , it follows from the above lemma that  $\text{char}_R(M) \subseteq \text{char}_R(N)$  if and only if  $\text{char}_R(M)_P \subseteq \text{char}_R(N)_P$  for every height-one prime  $P \in \text{Supp}(M) \cup \text{Supp}(N)$ .

*Example 9.5.* Suppose  $I$  is reflexive and let  $\mathfrak{a} \subseteq R$  be such that  $R/\mathfrak{a}$  is a normal domain. Then  $I(R/\mathfrak{a})$  need not be reflexive. For a general ideal  $I \subseteq R$ , it can happen that

$$(I(R/\mathfrak{a}))^{**} \neq (I^{**}(R/\mathfrak{a}))^{**}.$$

Here,  $I^{**}$  is the reflexive closure with respect to  $R$  and  $(I(R/\mathfrak{a}))^{**}$  is the reflexive closure with respect to  $R/\mathfrak{a}$ . Let us take a look at the following simple example. Take  $R = \mathbb{Z}_p[[x, y]]$ ,  $I = (x, y)$ , and  $\mathfrak{a} = (x)$ . Then  $(I^{**}(R/\mathfrak{a}))^{**} = R/xR$ , since there is no height-one prime of  $R$  containing  $I$ . But  $(I(R/\mathfrak{a}))^{**} = y(R/xR)$ .

**Proposition 9.6.** *Let  $R$  be a Noetherian normal domain. Then the following hold.*

- (1) *Let  $M$  be a finitely generated torsion  $R$ -module. Then we have*

$$\text{char}_R(M) = \left( \prod_{\text{ht } \mathfrak{p}=1} \mathfrak{p}^{\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})} \right)^{**} = \text{Fitt}_R(M)^{**}.$$

*In particular,  $\text{Fitt}_R(M) \subseteq \text{char}_R(M)$  and if  $R$  is a UFD, then*

$$\text{Fitt}_R(M) \subseteq \prod_{\text{ht } \mathfrak{p}=1} \mathfrak{p}^{\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})} = \text{char}_R(M).$$

- (2) *Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be a short exact sequence of finitely generated torsion  $R$ -modules. Then*

$$\text{char}_R(M) = (\text{char}_R(L) \cdot \text{char}_R(N))^{**}.$$

*Proof.* (1): Since the characteristic ideal is reflexive, the first equality follows by taking localization at all height-one primes of  $R$ . The second equality follows from the fact

$$\text{Fitt}_{R_P}(M_P) = (PR_P)^{\ell_{R_P}(M_P)}$$

for any height-one prime ideal  $P \subseteq R$ . The second assertion is due to the fact that a height-one prime in a UFD is principal.

(2): This is clear, since the length is additive with respect to short exact sequences.  $\square$

*Example 9.7.* The ordinary power of a height-one prime in a normal domain is not necessarily reflexive. Here is an example. Let  $R = \mathbb{Z}_p[[x^2, xy, y^2]]$  and let  $\mathfrak{p} = (x^2, xy)$ . Then  $R$  is a normal domain and  $\text{ht}(\mathfrak{p}) = 1$ . Then  $\mathfrak{p}^2 = (x^4, x^3y, x^2y^2)$  and  $\mathfrak{p}^2 \neq \mathfrak{p}^{(2)}$ . In fact,  $x^2 \notin \mathfrak{p}^2$ , but  $\mathfrak{p}^{(2)} = (x^2)$ .

Now let  $\mathfrak{q} = (xy, y^2)$  and  $M = R/(\mathfrak{p} \cap \mathfrak{q})$ , which is torsion over  $R$ . Then one verifies that

$$\text{Fitt}_R(M) = \mathfrak{p} \cap \mathfrak{q} \not\subseteq \prod_{\text{ht } \mathfrak{p}=1} \mathfrak{p}^{\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})} = \mathfrak{p}\mathfrak{q},$$

which tell us that Proposition 9.6 (1) is the most optimal.

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